Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 4 — Solutions

1. Chapter 2, Problem 1
2. Chapter 2, Problem 2
3. Chapter 2, Problem 3
4. Chapter 2, Problem 5
5. Chapter 2, Problem 6
6. Chapter 2, Problem 7b) (we did a) in class)
7. Chapter 2, Problem 10

   Solutions: See the supplementary solutions page.

8. You are in a room with 5 doors, and your host tells you that behind each of two randomly chosen doors he has placed a prize (all choices of two doors equally likely). You open the doors, one after another, from left to right. Let X be the random variable that measures the number of doors you open after seeing the first prize you see, but before seeing the second prize.

   (a) What are the possible values that X can take on?

   Solution: $0, 1, 2$ or 3

   (b) Compute the probability mass function of X.

   Solution: There are 120 possible orders for opening the doors, all equally likely. Number in which prize doors are 1st and 2nd: $2!3! = 12$; number in which prize doors are 2nd and 3rd: $2!3! = 12$; number in which prize doors are 3rd and 4th: $2!3! = 12$; number in which prize doors are 4th and 5th: $2!3! = 12$; so number in which $X = 0$: 48.

   Number in which prize doors are 1st and 3th: $2!3! = 12$; number in which prize doors are 2nd and 4th: $2!3! = 12$; number in which prize doors are 3rd and 5th: $2!3! = 12$; so number in which $X = 1$: 36.

   Number in which prize doors are 1st and 4th: $2!3! = 12$; number in which prize doors are 2nd and 5th: $2!3! = 12$; so number in which $X = 2$: 24.

Number in which prize doors are 1st and 5th: $2!3! = 12$; so number in which $X = 3$: 12.

So

$$p_X(x) = \begin{cases} \frac{48}{120} = .4 & \text{if } x = 0 \\ \frac{36}{120} = .3 & \text{if } x = 1 \\ \frac{24}{120} = .2 & \text{if } x = 2 \\ \frac{12}{120} = .1 & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

(Note that these numbers add to 1.)

(c) What is the probability that X is at least 2?

Solution: $\Pr(X \geq 2) = .1 + .2 = .3$.

9. Consider the following seven random processes:

(a) I buy a snack at the Huddle every day, and I eat the snack the same day as I buy it 70% of the time. Let X be the number of times, within a four week period, that I eat my snack on the day I buy it.

(b) 10% of milk cartons at Martins Supermarket are fat-free, 20% are 1% milk, 55% are 2% milk, and the rest are full-fat (3.5% fat). I pick 10 cartons at random. Each time, I record "0", "1", "2" or "3.5", depending on whether I chose fat-free, 1%, 2% or full-fat. Let X be the total of my 10 readings.

(c) I watch people walking into class, until I see the first person who is 6 foot tall or taller. Let X be the number of people I have to watch for this to happen.

(d) A magician asks six people, one after the other, to take a card from an ordinary deck of cards. After each person selects a card, they replace it in the deck. After this is done, he guesses the card that each person choose. Let X be the number of cards he guesses correctly.

(e) Same as above, but now people do not return their cards.

(f) When the Yankees play the Red Sox in NYC, they win with probability .7. When they play in Boston, they win with probability .3. All games are independent. Let X be the number of games the Yankees win, if they play 10 games with the Red Sox in Boston, and 8 in NYC.

(g) Same as above, but now X is the number of games that the home team wins (Red Sox play home games in Boston, Yankees in NYC).

For each of the seven random variables X's, say whether it is binomial or not. In those cases where the random variable is binomial, say what n and p are; in those cases where the random variable is not binomial, explain why not.

Solution: a) Binomial$(28, .7)$. b) Not binomial: the individual trials are independent, but I'm not counting "success" or "failure" for them. c) Not binomial: I don't fix number of trials in advance. d) Assuming the magician is guessing randomly, Binomial$(52, 1/52)$. e) Assuming the magician does not guess the same card for two different people (which wouldn't make much sense in this context), it's not binomial, because the trials are not independent, or have same success probabilities. f) Not binomial: the trials don't all have the same success probability. g) Binomial$(18, .7)$.

10. When a passanger books a flight on an airplane, (s)he will actual go ahead and use the ticket only 96% of the time. All passengers operate independently of each other. A plane has 210 seats. How many seats can the airline sell for this plane, and still be 95% sure that there will be enough seats for all the passengers who show up?

    Solution: If he airline sells n tickets, then the number of people who show up is Binomial with parameters n and .96, and there are enough seats if this binomial takes value 210 or less. So we are looking for the largest n such that

    $$\Pr(X \leq 210) \geq .95, \qquad X \sim \text{Binomial}(n, .96).$$

    Using a binomial calculator, we find that $n = 214$.

11. 10 people each line up at a craps table, and each (independently) rolls two dice. The house calls this process a "success" if the number of people who roll a sum of either 7 or 11 is four or fewer.

    (a) When this experiment is performed, what's the probability that the house will record a success?

    Solution: Let X be the number from among the ten who roll either 7 or 11. X is binomial with $n = 10$, $p = (6 + 2)/36 \approx .222$, so $\Pr(X \leq 4) \approx .95$. This is the probability that the house records "success".

    (b) If this trial is repeated many times, what's the probability that it takes between 3 and 5 repeats (inclusive) for the first success to be recorded?

    Solution: Let Y be the number of times the trial is repeated until the first success is recorded. Y is a geometric random variable, with $p = .95$ (from the previous part). So

    $$\Pr(3 \leq Y \leq 5) = (.05)^2(.95) + (.05)^3(.95) + (.05)^4(.95) \approx .0025.$$

12. 50,000 sensors are dropped randomly onto a desert surface that is shaped like a square, with dimensions 200 meters by 200 meters. I examine a particular 1 meter by 1 meter piece of the surface. Use the Poisson random variable to estimate:

    (a) the probability that the piece has at least one sensor in it.

    Solution: Let X be the number of sensors found in the 1 by 1 region. This is $1/40,000$th of the whole area, so the expected number of sensors is $50,000/40,000 = 1.25$; we use a Poisson random variable with $\lambda = 1.25$ to model X.

    $$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - e^{-1.25} \approx .713.$$

(b) the probability that it has either 1 or 2 sensors in it.

Solution:

$$\Pr(X \in \{1,2\}) = 1.25e^{-1.25} + \frac{(1.25)^2}{2!}e^{-1.25} \approx .582.$$

13. Let X be a Poisson random variable with parameter λ. Show that

$$\Pr(X \text{ is even}) = \frac{1}{2}\left(1+e^{-2\lambda}\right).$$

(**Hint**: think about the Taylor expansion of $e^{-\lambda}+e^{\lambda}$ about $\lambda = 0$.)

Solution:

$$\Pr(X \text{ is even}) = \Pr(X=0)+\Pr(X=2)+\Pr(X=4)+\ldots = e^{-\lambda}+\frac{\lambda^2}{2}e^{-\lambda}+\frac{\lambda^4}{4!}e^{-\lambda}+\ldots.$$

The Taylor series of e^{λ} is

$$e^{\lambda} = 1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\frac{\lambda^4}{4!}+\ldots$$

and the Taylor series of $e^{-\lambda}$ is

$$e^{-\lambda} = 1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\frac{\lambda^4}{4!}+\ldots,$$

so the Taylor series of $(e^{\lambda}+e^{-\lambda})/2$ is

$$e^{\lambda} = 1+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\ldots.$$

So

$$\Pr(X \text{ is even}) = e^{-\lambda}\left(1+\frac{\lambda^2}{2}+\frac{\lambda^4}{4!}+\ldots\right) = e^{-\lambda}\left(\frac{e^{\lambda}+e^{-\lambda}}{2}\right) = \frac{1}{2}\left(1+e^{-2\lambda}\right).$$

14. If I toss a coin repeatedly, the probability that after, say, 10 tosses, I haven't seen the first head, is $(1/2)^{10}$. If I tell you that I already tossed the coin 6 times, and still haven't seen a head, then the probability that after a *further* 10 tosses I *still* haven't seen the first head is *still* $(1/2)^{10}$ — the coin has no "memory" of the first 6 tosses; probability-wise, you might as well reset the counter after learning about the 6 failures. The way to encode this common sense idea is this: if X is a geometric random variable that counts the number of tosses until the first head, then $\Pr(X > 16|X > 6) = \Pr(X > 10)$ (the probability that it takes me more than 16, given that it's already taken me more than 6, is exactly the same as the probability that it takes me more than 10, with me starting counting from the moment that the conditional information about the first 6 tosses is given).

(a) Verify this "memorylessness" in general: show that if $X \sim \text{Geometric}(p)$ then for any integers m and k, $\Pr(X > m + k | X > k) = \Pr(X > m)$.

Solution:

$$\Pr(X > m + k | X > k) = \frac{\Pr(X > m + k \ AND \ X > k)}{\Pr(X > k)} =$$

$$\frac{\Pr(X > m + k)}{\Pr(X > k)} = \frac{(1-p)^{m+k}}{(1-p)^k} = (1-p)^m = \Pr(X > m).$$

(b) Show, by an example, that if X is a binomial random variable (for concreteness, the number of sixes you see when you roll a dice 10 times), then it is not the case that $\Pr(X > m + k | X > k) = \Pr(X > m)$.

Solution: Take $m = 1$ and $k = 1$, for example (basically any choice will do).

$$\Pr(X > 1 + 1 | X > 1) = \frac{\Pr(X > 2 \ AND \ X > 1)}{\Pr(X > 1)} =$$

$$\frac{\Pr(X > 2)}{\Pr(X > 1)} \approx \frac{.2249}{.5156} \approx .4362 \neq .5156 \approx \Pr(X > 1).$$

(c) Let X be a random variable that takes values $0, 1, 2, 3, \ldots$, and that satisfies the memorylessness property $\Pr(X > m + k | X > k) = \Pr(X > m)$ for all m and k. Show that X *must* be a geometric random variable.

Solution: Notice that

$$\Pr(X > m + k | X > k) = \frac{\Pr(X > m + k \ AND \ X > k)}{\Pr(X > k)} = \frac{\Pr(X > m + k)}{\Pr(X > k)}$$

so $\Pr(X > m + k | X > k) = \Pr(X > m)$ is the same as

$$\Pr(X > m + k) = \Pr(X > m) \Pr(X > k).$$

Plugging in $m = 1$ and $k = 0$ we get

$$\Pr(X > 1) = \Pr(X > 1) \Pr(X > 0),$$

so $\Pr(X > 0) = 1$, that is, $\Pr(X = 0) = 0$. So we can in fact assume that X takes values $1, 2, \ldots$.
With $m = k = 1$, get

$$\Pr(X > 2) = \Pr(X > 1)^2.$$

With $m = 2, k = 1$, get

$$\Pr(X > 3) = \Pr(X > 2) \Pr(X > 1),$$

which together with $\Pr(X > 2) = \Pr(X > 1)^2$ gives

$$\Pr(X > 3) = \Pr(X > 1)^3.$$

Continuing in this way, we get that for all $\ell > 0$,

$$\Pr(X > \ell) = \Pr(X > 1)^\ell.$$

Define p by $1 - p = \Pr(X > 1)$, so $\Pr(X > \ell) = (1-p)^\ell$. Then for $\ell \geq 2$

$$\Pr(X = \ell) = \Pr(X > \ell - 1) - \Pr(X > \ell) = (1-p)^{\ell-1} - (1-p)^\ell = p(1-p)^{\ell-1}.$$

Here's what we've shown: there is some number p, such that for each $\ell \geq 2$,

$$\Pr(X = \ell) = p(1-p)^{\ell-1}.$$

But $\Pr(X > 1) = 1 - p$, and $\Pr(X = 0) = 0$, so $\Pr(X = 1) = p$, so the above formula works for $\ell = 1$ as well. The mass function of X is:

$$p_X(x) = \begin{cases} p(1-p)^{x-1} & \text{if } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

This is exactly the mass function of a Geometric with parameter p.