The basic rules of counting

Math 30530, Fall 2012

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Basic counting rule 1 — The sum rule

- **Sum rule 1**: if an experiment can proceed in one of two ways, with
  - n_1 outcomes for the first way, and
  - n_2 outcomes for the second,

  then the total number of outcomes for the experiment is

$$n_1 + n_2$$

  Example: Movie *or* dinner? $\#$(screens) $+$ $\#$(restaurants)

- **Sum rule 2**: if an experiment can proceed in one of m ways, with
  - n_1 outcomes for the first way,
  - n_2 outcomes for the second, $\ldots$, and
  - n_m outcomes for the mth,

  then the total number of outcomes for the experiment is

$$n_1 + n_2 + \ldots + n_m$$

Basic counting rule 2 — The product rule

- **Product rule 1**: if an experiment is performed in two stages, with
  - n_1 outcomes for the first stage, and
  - n_2 outcomes for the second, *REGARDLESS OF FIRST*,

  then the total number of outcomes for the experiment is

  $$n_1 n_2$$

  Example: Movie *and* dinner? $\#$(screens) $\times$ $\#$(restaurants)

- **Product rule 2**: if an experiment is performed in m stages, with
  - n_1 outcomes for the first stage, and
  - n_2 outcomes for the second, *REGARDLESS OF FIRST*, . . ., and
  - n_m outcomes for the mth, *REGARDLESS OF ALL PREVIOUS*,

  then the total number of outcomes for the experiment is

  $$n_1 n_2 \ldots n_m$$

Selecting r items from n, WITHOUT REPLACEMENT

- **Order matters**:

$$n(n-1)\dots(n-(r-1)) = \frac{n!}{(n-r)!}$$

Example: 1st, 2nd and 3th in race with 8 runners? $8 \times 7 \times 6 = 336$

- **Order doesn't matter**:

$$\frac{n(n-1)\dots(n-(r-1))}{r!} = \binom{n}{r}$$

Example: Top three in race with eight runners? $\binom{8}{3}$

Basic counting rule 3 — The overcount rule

- If x is an initial count of some set of objects, and each object you want to count appears y times in x, then the correct count is

$$x/y$$

Some examples

- How many ways can 10 people form a committee of 6, with a chair, a secretary a treasurer and 3 general members?

$$\binom{10}{6} 6.5.4 = 25200$$

- What if John and Pat will not serve together, and Pat will only serve if Ellen is the chair?
  First consider committees with John (so without Pat), then those with Pat (so also with Ellen as chair, and without John), and then those without both John and Pat

$$\binom{8}{5} 6.5.4 + \binom{7}{4} 5.4 + \binom{8}{6} 6.5.4 = 10780$$

- How many 9 digit numbers can be formed with four 1's, three 2's and two 3's?

$$\frac{9!}{4!3!2!} = 1260$$

Selecting r items from n, WITH REPLACEMENT

- **Order matters**:

$$n^r$$

Example: 8 digit numbers with prime digits? 4^8

Example: In a group of 23 people, how likely is it that they all have their birthdays on a different date?

$$\frac{\text{ways of choosing 23 different dates, in order}}{\text{ways of choosing 23 dates, in order}}$$
$$= \frac{365 \times 364 \times \ldots \times 343}{365 \times 365 \times \ldots \times 365}$$
$$\approx .4927 < 50\%$$

Selecting r items from n, WITH REPLACEMENT

- **Order doesn't matter**:

$$\binom{n+r-1}{r}$$

Example: Select 8 single-digit primes, no particular order?

$$\binom{11}{8} = 165$$

Some examples

- I have 36 identical prizes to distribute to the class (53 people). All I care about is how many prizes each student gets. How many possible ways to distribute are there?

  Here $n = 53$ (I'm choosing from pool of students), $r = 36$ (I'm choosing students to give prizes to), and I'm choosing with replacement, order not mattering, so solution is

$$\binom{53+36-1}{36} = \binom{88}{36} \approx 6 \times 10^{34}$$

- What's the probability that Zeke doesn't get a prize, assuming all ways of distribution equally likely?

$$\binom{87}{36} \Big/ \binom{88}{36} \approx .59$$

- On average, how many people don't get coupons?

$$53 \times .59 \approx 31.3$$

Summary of counting problems

Selecting r items from n, WITHOUT REPLACEMENT

- ORDER MATTERS:
$$\frac{n!}{(n-r)!}$$
- ORDER DOESN'T MATTER:
$$\binom{n}{r}$$

Selecting r items from n, WITH REPLACEMENT

- ORDER MATTERS:
$$n^r$$
- ORDER DOESN'T MATTER:
$$\binom{n+r-1}{r}$$