Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 7 — Solutions

1. **GW** 25.1: The area of the triangle is 9/2, so on the triangle the joint density is 2/9; everywhere else it is 0. The region R of the triangle where $X+Y>2$ is shown shaded in figure 1 of the figures page, and $\Pr(X+Y\geq 2)$ is the double integral of the density over that region. Since the density is constant, this is just 2/9 times the area of the region, or $(2/9)\times(5/2)=5/9$.

2. **GW** 25.2: As with the previous question, we calculate the area of the shaded region in figure 2 of the figures page, divided by the total area; it's 12/18 or 2/3.

3. **GW** 25.8: a) $\int_1^2\int_3^4 \frac{1}{304}(x+1)(y^2+1)\ dydx = \frac{25}{228}\approx .109649$.
   b)
$$f_X(x)=\begin{cases} 0 & \text{if } x<0 \text{ or } x>4 \\ \frac{1}{12}(x+1) & \text{if } 0\leq x\leq 4. \end{cases}$$
   c)
$$f_Y(y)=\begin{cases} 0 & \text{if } y<0 \text{ or } y>4 \\ \frac{3}{76}(y^2+1) & \text{if } 0\leq y\leq 4. \end{cases}$$

4. **GW** 25.20: We have to do a double integral over the shaded region shown in figure 3 of the figures page, of the joint density. This is best done as an iterated integral with x on the outside:
$$\int_{x=0}^{1/2}\int_{y=-1}^{2x}\left(\frac{1}{4}\cos x\sin y+\frac{1}{4}\right)\ dxdy=\frac{1}{48}\left(9-12\sin(1/2)+4\sin(3/2)\right)\approx .150768.$$

5. **GW** 26.4: The marginal density of X is
$$f_X(x)=\begin{cases} 0 & \text{if } x<0 \text{ or } x>1 \\ \int_0^1 \frac{12}{7}(xy+x^2)\ dy=\frac{6x}{7}+\frac{12x^2}{7} & \text{if } 0\leq x\leq 1. \end{cases}$$
   The marginal density of Y is
$$f_Y(y)=\begin{cases} 0 & \text{if } y<0 \text{ or } y>1 \\ \int_0^1 \frac{12}{7}(xy+x^2)\ dx=\frac{4}{7}+\frac{6y}{7} & \text{if } 0\leq y\leq 1. \end{cases}$$
   The product of these marginals is not the original joint density, so they are *not* independent.

6. **GW** 26.7: a) As we will see below, $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$, so these random variables are *not* independent.

   b) The marginal density of X is

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 10 \\ \int_0^{10-x} \frac{3xy}{1250}\, dy = \frac{3x(x-10)^2}{1250} & \text{if } 0 \leq x \leq 10. \end{cases}$$

   The marginal density of Y is

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } y > 10 \\ \int_0^{10-y} \frac{3xy}{1250}\, dx = \frac{3y(y-10)^2}{1250} & \text{if } 0 \leq y \leq 10. \end{cases}$$

7. **GW** 26.8: We want $\Pr(X < Y)$. Assuming independence (otherwise we have no chance to solve the problem), the joint density of X and Y is

$$f_{X,Y}(x,y) = 120e^{-12x-10y}$$

   (if $x, y \geq 0$; it's 0 otherwise).

   $\Pr(X < Y)$ is the double integral of this density function over the shaded region in figure 4 of the figures page, i.e.

$$\Pr(X < Y) = \int_{x=0}^{\infty} \int_{y=x}^{\infty} 120e^{-12x-10y}\, dydx = \frac{6}{11} \approx .5454.$$

8. **GW** 26.12: a) On the triangle (area 8) the joint density is 1/8. The probability that $X < 3$, $Y < 3$ is the area of the shaded region in figure 5 of the figures page, divided by 8, i.e. 7/8.

9. **GW** 28.1: The marginal density of X is

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 3 \\ \int_{y=0}^{3-x} \frac{2}{9}\, dy = \frac{2(3-x)}{9} & \text{if } 0 \leq x \leq 3. \end{cases}$$

   So $E(X) = \int_0^3 x \frac{2(3-x)}{9}\, dx = 1$.

10. **GW** 28.13: $E(X) = \int_0^5 \frac{3x}{245}(x^2+8)\, dx = \frac{615}{196} \approx 3.13776$.

11. **GW** 28.21: The marginal density of X is

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 2 \\ \int_{y=0}^{4} \frac{3}{80}(x^2+y)\, dy = \frac{3}{10} + \frac{3x^2}{20} & \text{if } 0 \leq x \leq 2. \end{cases}$$

   So $E(X) = \int_0^2 x \left(\frac{3}{10} + \frac{3x^2}{20}\right)\, dx = \frac{6}{5} = 1.2$.

12. (a) **28.22** with **one** child:

Let D be the point of the perimeter of the rink where the door is located. Let D' be the point opposite the door. Imagine breaking the perimeter at D', and rolling it out as a straight line on the x-axis with D at the origin (so it stretches from -50π to 50π). Let X be the position of the child when the mother arrives; X is uniform on $[-50\pi, 50\pi]$, so has density:

$$f_X(x) = \begin{cases} 0 & \text{if } x < -50\pi \text{ or } x > 50\pi \\ \frac{1}{100\pi} & \text{if } -50\pi \le x \le 50\pi. \end{cases}$$

The mother is interested in the distance from her to the child. This is $Z = |X|$. The CDF of Z is given by

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \Pr(Z \le z) = \Pr(-z \le X \le z) = \frac{2z}{100\pi} & \text{if } 0 \le z \le 50\pi \\ 1 & \text{if } z > 50\pi. \end{cases}$$

The density of Z is given by differentiating the CDF:

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \text{ or } z > 50\pi \\ \frac{1}{50\pi} & \text{if } 0 \le z \le 50\pi. \end{cases}$$

So $E(Z) = \int_0^{50\pi} \frac{z}{50\pi}\, dz = 25\pi$.

(b) **28.22** with **two** children:

As before let D be the point of the perimeter of the rink where the door is located. Let D' be the point opposite the door. Imagine breaking the perimeter at D', and rolling it out as a straight line on the x-axis with D at the origin (so it stretches from -50π to 50π). Let X be the position of the first child when the mother arrives and Y the position of the second; X, Y are uniform on $[-50\pi, 50\pi]$, and independent, so have joint density:

$$f_{X,Y}(x, y) = \begin{cases} 0 & \text{if } x < -50\pi \text{ or } x > 50\pi \text{ or } y < -50\pi \text{ or } y > 50\pi \\ \frac{1}{(100\pi)^2} & \text{if } -50\pi \le x, y \le 50\pi. \end{cases}$$

The mother is interested in the distance from her to the nearest child. This is $Z = \min\{|X|, |Y|\}$. The CDF of Z is given by

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \Pr(Z \le z) = \Pr(\min |X|, |Y| \le z) & \text{if } 0 \le z \le 50\pi \\ 1 & \text{if } z > 50\pi. \end{cases}$$

To calculate $\Pr(\min |X|, |Y| \le z)$, note that the event we are looking at is the union of $|X| \le z$ and $|Y| \le z$ (one or the other), and so

$$\begin{aligned} \Pr(\min |X|, |Y| \le z) &= \Pr(|X| \le z) + \Pr(|Y| \le z) - \Pr(|X| \le z, |Y| \le z) \\ &= \frac{2z}{100\pi} + \frac{2z}{100\pi} - \left(\frac{2z}{100\pi}\right)^2 \end{aligned}$$

(the very last part because X, Y are independent). So the density of Z, which is given by differentiating the CDF, is:

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \text{ or } z > 50\pi \\ \frac{1}{25\pi} - \frac{2z}{(50\pi)^2} & \text{if } 0 \leq z \leq 50\pi. \end{cases}$$

So $E(Z) = \int_0^{50\pi} z \left(\frac{1}{25\pi} - \frac{2z}{(50\pi)^2} \right) dz = (50\pi)/3$.

13. The following density is a special case of one that occurs fairly commonly in economics and social science. It's called the *Pareto* or *Zipf* density (Wikipedia has good pages on both). There's a Pareto density for each $\alpha > 1$, and it's given by

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{c}{x^\alpha} & \text{if } x \geq 1. \end{cases}$$

Here c is a constant that depends on α.

(a) For each $\alpha > 1$, find the value of $c = c(\alpha)$ that makes f_α a valid density:

$$\int_1^\infty \frac{c}{x^\alpha}\, dx = \left[\frac{c}{(1-\alpha)x^{\alpha-1}} \right]_{x=1}^\infty = \frac{c}{\alpha - 1}.$$

So we must take $c = \alpha - 1$.

(b) For which values of α does the density f_α have a finite expectation:

$$E(X) = \int_1^\infty x \frac{\alpha - 1}{x^\alpha}\, dx = \int_1^\infty \frac{\alpha - 1}{x^{\alpha - 1}}\, dx = \left[\frac{\alpha - 1}{(2-\alpha)x^{\alpha-2}} \right]_{x=1}^\infty.$$

If $\alpha > 2$ then this integral converges (to $\frac{\alpha-1}{\alpha-2}$). If $\alpha \leq 2$ then it diverges. So there is finite expectation for $\alpha > 2$.