Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 5 — Solutions

1. **GW** 15.7

   a) If X is number of draws, then $X \sim \text{Geometric}(4/52)$, so $E(X) = 52/4 = 13$.

   b) Taking at least four is the same as failing on the first three trials, so the probability is $(12/13)^3$. Alternately, its

   $$(12/13)^3(1/13) + (12/13)^4(1/13) + \ldots = \frac{(12/13)^3(1/13)}{1-(12/13)} = (12/13)^3.$$

   c) Let Y be number of weeks in six weeks in which it takes at least 4 cards; $Y \sim \text{Binomial}(6, (12/13)^3)$, so $\Pr(Y = 0) = ((12/13)^3)^6 = (12/13)^{18}$. Here we are using a binomial random variable.

2. **GW** 15.8

   a) $X \sim \text{Geoemetric}(.07)$, so $E(X)$, the expected number of trials until success, is $1/.07 \approx 14.28$.

   b) $\Pr(X \geq 20) = (.93)^{19} \approx .25$ (failure on each of first 19 trials)

   c) Let Y be the number of questions on which he needs at least 20 tries; Y is binomial with parameters $n = 20$, $p = (.93)^{19}$, so the probability that on each question it takes qt least 20 tries is $((.93)^{17})^{10} = (.93)^{170}$.

   d) Let T be time to get a question right; $T = 2 + X/2$ (measuring in minutes), so (by linearity) $E(T) = 2 + E(X)/2 \approx 9.28$. This seems pretty fast ... maybe he is indeed better off guessing ...

3. **GW** 15.20

   a) The fact that he has had to check 4 already is irrelevant, by memorylessness. So the probability of having to check at least 5 more, is just the initial probability of having to check at least 5, which is $(.98)^4$ (need 4 failures in a row initially).

   b) Again, by memorylessness, the first four trails are irrelevant, and the probability is .02.

   c) .02.

   d) $1/.02 = 50$.

   e) $C = 100 + 10X$ is his cost; $E(C) = 100 + 10E(X) = 5,100$.

4. **GW** 15.22

$$\begin{aligned}\Pr(X = Y) &= \sum_{i=1}^{\infty} \Pr(X = i, Y = i) \\ &= \sum_{i=1}^{\infty} q_1^{i-1} p_1 q_2^{i-1} p_2 \\ &= \frac{p_1 p_2}{1 - q_1 q_2}.\end{aligned}$$

5. We said in class that the geometric random variable is the only memoryless discrete random variable. This exercise ask you to show that the binomial random variable is not in general memoryless.

   - Let $X \sim$ Binomial(10, .2). Compute $\Pr(X > 6|X > 2)$ and $\Pr(X > 4)$, and verify that the answers you get are different.

$$\Pr(X > 6|X > 2) = \frac{\Pr(X > 6, X > 2)}{\Pr(X > 2)} = \frac{\Pr(X > 6)}{\Pr(X > 2)} = \frac{.0009}{.3222} = .00279 \neq .0328 = \Pr(X > 4).$$

   (Calculations done using online binomial calculator).

6. **GW** 16.4

   a) If X is number of people she needs to ask, X is Negative Binomial with $r = 5$, $p = .75$, so $E(X) = 20/3$.

   b) $\text{Var}(X) = qr/p^2 = 2.22...$, so standard deviation is square root of this, 1.49....

   c), d): I hadn't meant to assign these ... sorry!

7. **GW** 16.9

   a) If X is number of times you need to play, X is Negative Binomial with $r = 3$, $p = .25$. Playing for more than an hour is the same as $X > 12$. There are 2 ways to calculate this probability:

   1)

$$\Pr(X > 12) = 1 - \Pr(X \leq 12) = 1 - \sum_{x=3}^{12} \binom{x-1}{2} (.25)^2 (.75)^{x-2} = .3907.$$

   2) (Easier computation): $X > 12$ includes exactly the same outcomes as playing the game 12 times, and have 2 or fewer successes in those 12 tries, so

$$\Pr(X > 12) = \Pr(\text{Binomial}(12.25) \leq 2) = .3907.$$

   b) Number of minutes playing is $5X$. $E(5X) = 5E(X) = (5)(3)/(.25) = 60$.

8. **GW** 16.12

   a) Probability Callum wins in:

   Three games: $(.8)^3 = .512$

   Four games: $3(.8)^3(.2) = .3072$

   Five games: $6(.8)^3(.2)^2 = .12288$.

   b) Probability Philip wins in:

   Three games: $(.2)^3 = .008$

   Four games: $3(.2)^3(.8) = .0192$

   Five games: $6(.2)^3(.8)^2 = .03072$.

   c) $\Pr(C|5) = \Pr(C, 5)/\Pr(5) = .12288/(.12288 + .03072) = .8$ (of course - if it goes to five games, at one point they must have been tied 2-2, and Callum wins the last game with probability .8).

9. **GW** 21.1

   a) Here we don't care about the order in which the tickets are drawn, so number of ways is $\binom{30}{3} = 4060$.

   b) Now we care about order; there are $30 * 29 * 28 = 24360$ ways.

10. **GW** 21.3

    For all parts, the number of arrangements of the 11 letters (the denominator in the calculation) is $11!/(4!4!2!) = 34650$. We just need to calculate numerators.

    a) View "SSSS" as a new letter. Now we have a word with 8 letters: one M, two P's, four I's and one SSSS. There are $8!/(4!2!) = 840$ arrangements, so probability is

    $$\frac{840}{34650} = .024....$$

    b) Using the same reasoning as before, the numerator is now $7!/4! = 210$, so probability is

    $$\frac{210}{34650} = .0085....$$

    c) Build the word by first choose first letter (4 choices), then 5th (4 choices), then remainder (9! choices); numerator is $(4 * 4 * 9!)/(4!4!2!)$ (we have to deal with overcount). So probability is

    $$\frac{4 * 4 * 9!}{11!} = .145....$$

    (Notice that the denominators in both numerator and denominator cancel out).

d) 6 choices for first letter, 10! for rest, dealing with overcount get numerator of $6 * 10!/(4!4!2!)$, and so probability is

$$\frac{6 * 10!}{11!} = 6/11 = .54....$$

11. **GW** 21.10

a) There are nine ways: 8 women and 0 men, 7 women and 1 man, etc, all the way to 0 women and 8 men.

b) Now we need to specify man or women for each job, in order; there are 2^8 ways of doing this.

12. **GW** 21.17

Number of ways of making a selection of 8: $\binom{31}{8}$.

Number of ways of getting 6 red: $\binom{7}{6}\binom{24}{2}$.

Probability is

$$\frac{\binom{7}{6}\binom{24}{2}}{\binom{31}{8}} = .0002499....$$

13. **GW** 21.23

In both cases the denominator of the probability calculation will be 26^5.

a) Number of choices of distinct letters is $26 * 25 * 24 * 23 * 22$, so probability is

$$\frac{26 * 25 * 24 * 23 * 22}{26^5} = .6643....$$

b) For each way of selecting 5 distinct letters from 26, there is a unique way of assigning them to the five friends so that they are in ascending order; so the numerator here is $\binom{26}{5}$, and the probability is

$$\frac{\binom{26}{5}}{26^5} = .0055....$$

14. **GW** 21.28

a) $7 * 5 * 3 * 9 * 3 = 2835$.

b) $5 * 3 * 9 * 3 = 405$.

c) Now we are choosing $r = 3$ items from $n = 20$, with replacement, order not mattering, so the number of choices for toppings is $\binom{n+r-1}{r} = \binom{22}{3} = 1540$. If you are also allowed to choose a bagel, there are $7 * 1540 = 10780$ possibilities.

Here I'm assuming that "dressing" is a topping, since the question refers earlier to "20 toppings". A bagel with triple french dressing doesn't sound too appealing, though...

d) Now you simply have $\binom{20}{3} = 1140$ options for toppings, and (including bagel) 7980 total choices.

15. **GW** 21.28

    Oops. Repeat of last question...

16. **GW** 21.31

    There are two ways to block out 3 seats in a row for the family in the row of 4, and 4 ways in the row of 6, so 6 ways in all; there are 6 ways of arranging the members of the family together, and 7! ways of arranging everyone else, so $6 * 6 * 7! = 181,440$.

17. **GW** 21.35

    There are $\binom{33}{6}$ ways of choosing 6 socks. I'm assuming here that "sock" refers to a single sock - two socks together makes a pair.

    a) The number of ways of getting exactly 2 black, 2 white and 2 brown is

    $$\binom{21}{2}\binom{8}{2}\binom{4}{2},$$

    so the probability here is

    $$\frac{\binom{21}{2}\binom{8}{2}\binom{4}{2}}{\binom{33}{6}} = .03185....$$

    b) The number of ways of getting all white socks is $\binom{21}{6}$; the number of ways of getting all black socks is $\binom{8}{6}$; there is no way to get all brown socks. So the probability is

    $$\frac{\binom{21}{6} + \binom{8}{6}}{\binom{33}{6}} = .049....$$

    c) Number of ways of getting 2 whites, 4 blacks is $\binom{21}{2}\binom{8}{4}$; number of ways of getting 2 blacks, 4 whites is $\binom{21}{4}\binom{8}{2}$; doing this for all other pair-possibilities, get that the probability is

    $$\frac{\binom{21}{2}\binom{8}{4} + \binom{21}{4}\binom{8}{2} + \binom{21}{2}\binom{4}{4} + \binom{21}{4}\binom{4}{2} + \binom{8}{2}\binom{4}{4} + \binom{8}{4}\binom{4}{2}}{\binom{33}{6}} = .1975....$$