Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 2 — Solutions

1. For events A and B with $\Pr(A) > 0$, $\Pr(B) > 0$, we say that A is *positively correlated* with B if $\Pr(A|B) > \Pr(A)$ (in other words, B occurring *increases* the chance that A occurs). If $\Pr(A|B) < \Pr(A)$ then A is *negatively correlated* with B. (If $\Pr(A|B) = \Pr(A)$ then A is *not correlated* with B.)

   (a) Show that if A is positively correlated with B, then B is positively correlated with A; in other words, show that if $\Pr(A|B) > \Pr(A)$ then $\Pr(B|A) > \Pr(B)$. (The same holds for negatively correlated and uncorrelated; so we can from on just say "A and B are positively correlated", etc.)

   Solution: $\Pr(A|B) > \Pr(A)$ implies $\Pr(A \cap B)/\Pr(B) > \Pr(A)$ which implies $\Pr(A \cap B) > \Pr(A)\Pr(B)$ which implies $\Pr(A \cap B)/\Pr(A) > \Pr(B)$ which implies $\Pr(B|A) > \Pr(B)$.

   (b) Andy Murray and Roger Federer play a best-of-*five*-set tennis match. Each player has a .5 chance of winning each set, with each set independent of the others. Are the events "Federer wins the match (gets to three sets before Murray does)" and "Federer has a two sets to one advantage after three sets" positively correlated, negatively correlated, or uncorrelated?

   Solution: Let A be the event that Federer wins; $\Pr(A) = .5$ because the players are equally matched. Let B be the event that Federer has a two sets to one advantage after three sets. Given B, the sample space collapses to $\{f, mf, mm\}$, with probabilities 1/2, 1/4 and 1/4. The first two outcomes have Federer winning, so $\Pr(A|B) = .75 > \Pr(A)$, and so the events are *positively* correlated (naturally: Federer is more likely to win starting with a 2 sets to 1 advantage, then he is starting from scratch).

   (c) I roll two dice and note the numbers that come up. Are the events "The sum of the numbers coming up is even" and "The two numbers are within one of each other" positively correlated, negatively correlated, or uncorrelated?

   Solution: Let A be the event that the sum of the numbers is even. A contains the outcomes $11, 13, 22, 31, 15, 24, 33, 42, 51, 26, 25, 44, 53, 62, 46, 55, 64, 66$: 18 of 36 equally likely outcomes, so $\Pr(A) = .5$. Let B be the event that the two numbers are within one of each other. B contains the outcomes

$$11, 12, 21, 22, 23, 32, 33, 34, 43, 44, 45, 54, 55, 56, 65, 66:$$

16 equally likely outcomes, so conditioning on B occurring, each of these should get probability $1/16$. 6 of these 16 outcomes are in A, so $\Pr(A|B) = 6/16 = 3/8 < 1/2$. So the events are *negatively* correlated.

2. In one variant of the game of craps, a player rolls two dice. If the sum is 7 or 11, he wins immediately. If the sum is 2 or 12, he loses immediately. If he rolls anything else (3, 4, 5, 6, 8, 9, 10), then that becomes his "point", and he repeatedly rolls the pair of dice until EITHER he gets his point again, in which case he wins, OR he rolls 7 or 11, in which case he losses.

   Suppose a player's point is 5. What is the probability that he wins eventually? In other words, when rolling a pair of dice repeatedly, and each time noting the sum of the numbers that come up, what is the probability that the sum of 5 will come up before either 7 or 11? (**Hint**: Section 3.4 of the textbook!)

   Solution: The probability of rolling 5 as the sum is $4/36 = 1/9$, and the probability of rolling either 7 or 11 is $(6 + 2)/36 = 2/9$. In section 3.4, the following fact is proved: if you perform an experiment repeatedly until one of two disjoint events A or B occurs for the first time, then the probability that it is A that occurs for the first time is $\Pr(A)/(\Pr(A) + \Pr(B))$. In this case A is the event of rolling a 5, and B is the event of rolling 7 or 11 (disjoint events), so the probability that A occurs before B is $(1/9)/(1/9 + 2/9) = 1/3$.

3. Show that if $A_1, A_2, \ldots, A_n$ are independent events, then

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - (1 - \Pr(A_1))(1 - \Pr(A_2)) \ldots (1 - \Pr(A_n))$$

   Solution: Use DeMorgan's laws!

$$\begin{aligned}
\Pr(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - \Pr((A_1 \cup A_2 \cup \ldots \cup A_n)^c) \\
&= 1 - \Pr(A_1^c \cap A_2^c \cap \ldots \cap A_n^c) \\
&= 1 - \Pr(A_1^c)\Pr(A_2^c) \ldots \Pr(A_n^c) \quad \text{(using independence)} \\
&= 1 - (1 - \Pr(A_1))(1 - \Pr(A_2)) \ldots (1 - \Pr(A_n)).
\end{aligned}$$

4. ~~**GW** 4.1~~ (removed because the typos in the question are a little bit too much to correct without causing further confusion!)

5. **GW** 4.5

   Solution: a) Sample space has $15 + 10 = 25$ equally likely outcomes, of which $0 + 5$ are successes (green); so the probability is $5/25 = 1/5$.

   b) There are $10 + 15 + 5 = 30$ yellow boxes, of which 15 are brand B, so probability is $15/30 = 1/2$.

6. **GW** 4.6

   Solution: a) 12 of the 32 are either red or blue, so $12/32 = 3/8$.

   b) 4 of the 32 are sour, so $4/32 = 1/8$.

c) 28 of the skittles are not sour; this is the collapsed sample space for this conditional calculation, all points being equally likely. 4 of the non-sour skittles are purple, so the probability is $4/28 = 1/7$.

7. **GW** 4.8 (just a, b, c) In all cases, all points in the collapsed sample space are equally likely.

   Solution: a) 1/4 of all 7's are spades, so probability is 1/4.

   b) 13 of the 26 black cards are spades, so probability is $13/26 = 1/2$.

   c) 1 of he 13 spades is a 7, so probability is 1/13.

8. **GW** 4.10

   Solution: a) Let's use "Adam's law" or "the law of total probability":

$$\Pr(\text{salt}) = \Pr(\text{salt}|\text{campbells})\Pr(\text{campbells}) + \Pr(\text{salt}|\text{other})\Pr(\text{other}).$$

   So $\Pr(\text{salt}) = (18/72) * (72/138) + (1/3) * (66/138) = 40/138 = 20/69$.

   b) $\Pr(\text{campbells}|\text{salt}) = \Pr(\text{campbells and salt})/\Pr(\text{salt}) = (18/138)/(40/138) = 9/20$, just below .5, so you are more likely to get one made by another company.

9. **GW** 4.15 (for this question and the next, assume that when a baby is born (s)he is equally likely to be a boy or a girl, and the genders of a couples' children are independent)

   Solution: Just knowing that the couple has 2 children, there are four equally-likely outcomes to the experiment of observing the gender of the older followed by the gender of the younger: gg, gb, bg, bb. Knowing that at least one is a boy, the sample space collapses to 3 equally likely outcomes: gb, bg, bb. Two of these three outcomes has a child of each gender, so the probability is 2/3.

10. **GW** 4.16

    Solution: Just knowing that the couple has 3 children, there are eight equally-likely outcomes to the experiment of observing the gender of the oldest followed by the gender of the middle followed by the gender of the youngest: $ggg, ggb, gbg, bgg, gbb, bgb, bbg, bbb$. Knowing that they are not all girls, the sample space collapses to seven equally likely outcomes: $ggb, gbg, bgg, gbb, bgb, bbg, bbb$.

    a) 3 of the 7 outcomes have exactly one boy, so 3/7.

    b) 3 of the 7 outcomes have exactly two boys, so 3/7.

    c) 1 of the 7 outcomes has exactly three boys, so 1/7.

11. **GW** 4.17

    Solution: Referring to figure 3 on the figures page for the homework 1 solutions, we find that there are 12 (equally likely) outcomes in which Bob and Catherine are seated next to each other. Among these, there are 8 configurations in which Doug and Edna are seated next to each other. So the conditional probability is $8/12 = 2/3$.

12. **GW** 5.2

    Solution: H is the event of having an honors course; $\Pr(H) = .2$, $\Pr(H^c) = .8$. E is the event of enjoying semester; $\Pr(E|H) = .99$ and $\Pr(E|H^c) = .3$. By Bayes',

    $$\Pr(H|E) = \frac{\Pr(H \cap E)}{\Pr(E)} = \frac{\Pr(E|H)\Pr(H)}{\Pr(E|H)\Pr(H) + \Pr(E|H^c)\Pr(H^c)} = \frac{.99 * .2}{.99 * .2 + .3 * .8} = .452...$$

13. **GW** 5.9

    Solution: With the events having the obvious meanings,

    $$\Pr(C|H) = \frac{\Pr(C)\Pr(H|C)}{\Pr(H)} = \frac{.4 * .3}{.25} = .48.$$

    (No Bayes' formula here, really).

14. **GW** 5.11

    Solution: With the events having the obvious meanings,

    $$\Pr(Dab|For) = \frac{\Pr(For|Dab)\Pr(Dab)}{\Pr(For|Dab)\Pr(Dab) + \Pr(For|Prac)\Pr(Prac)} = \frac{.05 * .18}{.05 * .18 + .25 * .82} = .042...$$

15. **GW** 5.13

    Solution: Proportion of classes that are in chalkboard rooms that are Math or Stats: $.83 * (.75 + .1) = .7055$. Proportion of classes that are in dry-erase rooms that are Math or Stats: $.17 * (.08 + .27) = .0595$. So total percentage is 76.5%.

16. **GW** 5.19

    Solution: a)

    $$\Pr(H|A) = \frac{\Pr(H \cap A)}{\Pr(A)} = \frac{\Pr(H)\Pr(A|H)}{\Pr(H)\Pr(A|H) + \Pr(T)\Pr(A|T)} = \frac{.5 * (1/4)}{.5 * (1/4) + .5 * (1/6)} = 3/5.$$

    b)

    $$\Pr(T|A) = \frac{\Pr(T \cap A)}{\Pr(A)} = \frac{\Pr(T)\Pr(A|T)}{\Pr(H)\Pr(A|H) + \Pr(T)\Pr(A|T)} = \frac{.5 * (1/6)}{.5 * (1/4) + .5 * (1/6)} = 2/5.$$

17. **GW** 5.23

    Solution: The sample space initially has four outcomes, all equally likely: $\{gg, gb, bg, bb\}$ (first born listed first). The event C consists only of $\{gg, gb, bg\}$, so conditioning on this event we collapse to this sample space of size 3 with each outcome equally likely. D consists of one outcome, $\{gg\}$, which is entirely inside C, so $\Pr(D|C) = 1/3$.

18. **GW** 5.24

    Solution: There are 18 equally likely outcomes in which the blue dice has odd value: $11, 12, 13, 14, 15, 16, 31, 32, 33, 34, 35, 36, 51, 52, 53, 54, 55, 56$. (Blue dice listed first). Of these, only two have the sum exactly 4: $13, 31$. So the probability is $1/9$.

19. **GW** 5.28

Solution: Let C_n be the event that the coin comes up heads (for the first time) on the nth toss, and let C_∞ be the event that the coin never comes up heads. The C's form a partition of the sample space, so, writing E for the event that no one appears on any dice, we have:

$$\begin{aligned}
\Pr(E) &= \Pr(E \cap C_1) + \Pr(E \cap C_2) + \ldots + \Pr(E \cap C_n) + \ldots + \Pr(E \cap C_\infty) \\
&= \Pr(E|C_1)\Pr(C_1) + \Pr(E|C_2)\Pr(C_2) + \ldots + \Pr(E|C_n)\Pr(C_n) + \ldots + \Pr(E|C_\infty)\Pr(C_\infty).
\end{aligned}$$

By independence, $\Pr(C_n) = (1/2)^n$, and we calculated in class that $\Pr(C_\infty) = 0$. For each $n < infty$, $\Pr(E|C_n)$ is the event that we get no ones, on rolling n dice; that probability, again by independence, is $(5/6)^n$. So

$$\begin{aligned}
\Pr(E) &= (1/2)^1(5/6)^1 + (1/2)^2(5/6)^2 + \ldots + (1/2)^n(5/6)^n + \ldots + 0 \\
&= (5/12) + (5/12)^2 + \ldots + (5/12)^n + \ldots \\
&= \frac{(5/12)}{1-(5/12)} \\
&= \frac{5}{7}.
\end{aligned}$$

Here I used the geometric series formula:

$$a + ax + ax^2 + \ldots + ax^n + \ldots = \frac{a}{1-x}$$

with $a = x = 5/12$.