Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 1 — Solutions

1. A box contains four candy bars: two Mars bars, a Snickers and a Kit-Kat. I randomly draw a bar from the box and eat it, then draw a second and eat that, too. I record the type of bar I ate first, and the type I ate second.

   (a) List all the outcomes in the sample space of this experiment.
   Solution: $S = \{(M, M), (M, S), (M, K), (S, M), (S, K), (K, M), (K, S)\}$

   (b) Describe two different events in this experiment, *in words*.
   Solution: A: Both bars I eat are Mars bars; B: neither are Mars bars

   (c) For each of the two events you described in the last part, describe them by listing all their outcomes.
   Solution: $A = \{(M, M)\}; B = \{(S, K), (K, S)\}$

   (d) Are the two events you described mutually exclusive?
   Solution: No outcomes in common, so Yes.

   NB: There are many possible correct answers to this question

2. I have five keys in my pocket, each of which is a different colour (white, black, gold, silver, yellow). I repeatedly reach into my pocket and pull out a key at random, note its colour, and return it to my pocket. I keep doing this until I draw the gold key, at which point the experiment stops. (The gold key is the one I need to get into my house and out of the rain).

   (a) List all the outcomes in the sample space of this experiment.
   Solution: $S = \left\{(a_1, a_2, a_3, \ldots, a_n)|^{n\in\mathbb{N}^{>0}, \text{each } a_i\in\{\text{white,black,gold,silver,yellow}\},}_{\text{only } a_n=\text{gold}}\right\} \cup$
   $\{(a_1, a_2, a_3, \ldots)|\text{each } a_i \in \{\text{white, black, silver, yellow}\}\}$

   (b) Let E_n be the event that it takes me n or fewer trials to get the gold key. List all the outcomes in E_n. (You may need to use the set notation

   $$\{\text{things}|\text{these conditions are satisfied}\}$$

   here; see page 21.)
   Solution: $E_n = \left\{(a_1, a_2, a_3, \ldots, a_j)|^{j\le n, \text{each } a_i\in\{\text{white,black,gold,silver,yellow}\},}_{\text{only } a_n=\text{gold}}\right\}$

(c) Describe in words what it means for the event $(\cup_{n=1}^{\infty} E_n)^c$ to occur.

Solution: This is the event that I never select the gold key

(d) Are E_7 and E_8 mutually exclusive?

Solution: No, because for example the outcome (yellow, yellow, gold) is in both E_7 and E_8 (if I succeed at the third step, I have succeeded in 7 or fewer trials as well as in 8 or fewer)

3. A and B are mutually exclusive, with $\Pr(A) = .3$ and $\Pr(B) = .5$. What is the probability that

(a) either A or B occurs?

Solution: $\Pr(A \cup B) = \Pr(A) + \Pr(B) = .3 + .5 = .8$

(b) A occurs but not B?

Solution: If A occurs, then *automatically* B does not; so the probability that A occurs but not B is exactly the probability that A occurs; .3

(c) both A and B occur?

Solution: $\Pr(A \cap B) = \Pr(\emptyset) = 0$

4. A total of 28% of American men smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes.

(a) What percentage smoke neither cigarettes nor cigars?

Solution: Consider experiment of picking man at random, observing if they are cigarette smoker, cigar smoker. Let A be the event that they are cigarette smoker, and B the event that they are cigar smoker. We are given $\Pr(A) = .28$, $\Pr(B) = .07$ and $\Pr(A \cap B) = .05$. We want $\Pr((A \cup B)^c)$, which is:

$$\Pr((A \cup B)^c) = 1 - \Pr(A \cup B) = 1 - (\Pr(A) + \Pr(B) - \Pr(A \cap B)) = 1 - (.28 + .07 - .05) = .7$$

(b) What percentage smoke cigars but not cigarettes?

Solution: We want $\Pr(A \cap B^c)$, which is:

$$\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B) = .28 - .05 = .23$$

5. The dice in my right hand has three sides painted blue, one side painted red, and two sides painted green. The dice in my left hand has two sides painted blue, two sides painted red, and two sides painted green. I roll the two dice and record the two colours that come up (listing the right dice first).

(a) List the outcomes in the sample space.

Solution: $S = \{(B,B), (B,R), (B,G), (R,B), (R,R), (R,G), (G,B), (G,R), (G,G)\}$

(b) By using the fact that the outcomes of the two rolls are independent of each other, calculate the probabilities that i) the two dice both come up red, ii) the two dice show the same colour.

Solution: Probability of first dice coming up red is $1/6$, and probability for second dice is $1/3$, so by independence the probability of both coming up red is $1/18$.

By similar reasoning, the probability of both coming up blue is $1/6$ and the probability of both coming up green is $1/9$, so by disjointness the probability that both come up the same colour is $1/18 + 1/6 + 1/9 = 1/3$

6. Use a Venn diagram to explain why the following relations between events are always true:

   (a) $E \cap F \subset E \subset E \cup F$.

   Solution: See figure 1 on figures page. Note that "$E \cap F \subset E \subset E \cup F$" is really two separate statements: "$E \cap F \subset E$" and "$E \subset E \cup F$".

   (b) If $E \subset F$ then $F^c \subset E^c$.

   Solution: See figure 2 on figures page

7. Let E, F, G be three events. Using $\cup$, $\cap$ and ^{c}, find expressions for the following events:

   (a) Only E occurs.

   Solution: $E \cap F^c \cap G^c$

   (b) At least one of the events occurs.

   Solution: $E \cup F \cup G$

   (c) Both E and G, but not F, occur.

   Solution: $E \cap F^c \cap G$

   (d) Exactly two of the three events occur.

   Solution: $(E \cap F \cap G^c) \cup (E \cap F^c \cap G) \cup (E^c \cap F \cap G)$

   NB: There are many possible correct answers to this question

8. Show that for any events E and F, $\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1$.

   Solution: $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$, so $\Pr(E \cap F) = \Pr(E) + \Pr(F) - \Pr(E \cup F) \geq \Pr(E) + \Pr(F) - 1$, the last inequality since $\Pr(E \cup F) \leq 1$ and so $-\Pr(E \cup F) \geq -1$

9. Show that for any events A and B, the probability that exactly one of them occur is $\Pr(A) + \Pr(B) - 2\Pr(A \cap B)$.

   Solution: The probability that exactly one event occurs is

$$\begin{aligned} \Pr((A \cap B^c) \cup (A^c \cap B)) &= \Pr(A \cap B^c) + \Pr(A^c \cap B) \\ &= (\Pr(A) - \Pr(A \cap B)) + (\Pr(B) - \Pr(A \cap B)) \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B). \end{aligned}$$

10. **GW** 1.1 (there are many right answers to this one)

1) **Solution**: If we model the region inside the circle by $\{(x,y)|x^2+y^2<1\}$ (letting x-axis be north, y-axis east, center of circle at origin, units miles), then one possible outcome is $(0,0)$: the skydiver lands in the center of the circle

2) **Solution**: One possible event is $E=\{(x,y)|x^2+y^2\leq 1/4\}$, the event that the skydiver lands within a quarter mile of the center of the circle

3) **Solution**: $S=\{(x,y)|x^2+y^2<1\}$

11. **GW** 1.3

1) **Solution**: All we are asked to record is the number of draws Chris makes, so an outcome can be encode as a whole number greater than 0 (or the symbol ∞ indicating that he keeps drawing forever and never gets an orange). So one possible outcome is 3: it takes him three draws.

2) **Solution**: One possible event is $E=\{3,4,5\}$, the event that Chris takes at least three tries to get an orange.

3) **Solution**: $S=\{1,2,3,\ldots\}\cup\{\infty\}$

12. **GW** 1.8

a) **Solution**: Here we are not told specifically what to record as an outcome of the experiment, so this is left to our discretion. I decided to record the colors used, in the order that they were used. One possible approach is to label the three non-purple jars 1, 2, 3, and label the purple jar p. Then the sample space consists of all lists of distinct elements from $\{1,2,3,p\}$ that end with a p. So

$$S=\left\{\begin{array}{c}(p),(1,p),(2,p),(3,p),(1,2,p),(1,3,p),(2,1,p),(2,3,p),(3,1,p),(3,2,p),\\(1,2,3,p),(1,3,2,p),(2,1,3,p),(2,3,1,p),(3,1,2,p),(3,2,1,p)\end{array}\right\}$$

b) **Solution**: 16; see above for list

c) **Solution**: An event is a subset of the sample space. Here the sample space has 16 outcomes, so to build up a subset we have to make 16 independent decisions: is the first element in or out? the second? etc. So the number of possible subsets is $2\times 2\times\ldots\times 2=2^{16}=65536$

c) **Solution**: $(1,2,1,3,2,1,p)$; (p); $(2,2,2,2,2,2,2,p)$; $(1,2,3,p)$, for example

d) **Solution**: $S=\{(x_1,x_2,\ldots,x_n)|n$ a whole number, each $x_i\in\{1,2,3,p\}$, only $x_n=p\}\cup\{(x_1,x_2,\ldots)|$each $x_i\in\{1,2,3,f\}\}$

NB: There are many possible correct answers to this question. For example, one might simply record the number of colors used, in which case S is simply $\{1,2,3,4\}$.

13. **GW** 1.12

Solution: $S=\{(x,y)|x\geq 0,y\geq 0,x+y\leq 2\}$ (the diagonal line has equation $x+y=2$)

14. **GW** 2.3

Solution:

$$\begin{aligned}
\Pr(A \cup B \cup C) &= \Pr(A) + \Pr(B) + \Pr(C) \\
&\quad - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) \\
&\quad + \Pr(A \cap B \cap C) \\
&= .38 + .38 + .38 - .12 - .12 - .12 + .05 \\
&= .89
\end{aligned}$$

15. **GW** 2.11

    a) **Solution**: Each person has a 1/20 chance of winning, so my brother has a 19/20 chance of *not* winning

    b) **Solution**: Assuming that he is equally likely to finish in each of the 20 spots, he has a 17/20 chance of finishing out of the top 3.

    c) **Solution**: Again assuming that he is equally likely to finish in each of the 20 spots, he has a $10/20 = 1/2$ chance of finishing in the top 10.

16. **GW** 2.19

    a) **Solution**: $A \cup B \cup C$ contains outcomes 1, 3, 4, 6, so $4/6 = 2/3$ probability

    b) **Solution**: $A \cap B \cap C$ contains outcome 6 only, so 1/6 probability

17. **GW** 2.27

    Solution: You have two choices for first number, for each of these 10 choices of second, so $2 \times 10 = 20$ choices for the first two numbers $(22, 0; 32, 0; 22, 1; 32, 1;$ etc.). For each of these 20 choices, you have 46 choices for the third, so $20 \times 46 = 920$ in all. The worst case is that you have to try all 920.

18. **GW** 2.29

    a) **Solution**: 2 choices for first gender; for each choice, two choices for second, etc., so $2 \times 2 \times \ldots \times 2 = 2^7 = 128$ outcomes in all

    b) **Solution**: 6 outcomes with first female first

    $$(ffmmmmm, fmfmmmm, fmmfmmm, fmmmfmm, fmmmmfm, fmmmmmf),$$

    5 outcomes with first female second, 4 with first female third, 3 with first females fourth, 2 with first female fifth, 1 with first female sixth (and none with first female seventh). So $6 + 5 + 4 + 3 + 2 + 1 = 21$ outcomes in A_2.

    c) **Solution**: Doing an exhaustive list as in the last part, we find that there is 1 outcome in A_0, 7 in A_1, 21 in A_2, 35 in A_3, 35 in A_4, 21 in A_5, 7 in A_6 and 1 in A_7 (note that these all add to 128, the size of the sample space: there has to be some number of women)

    d) **Solution**: If each customer is equally likely to be a man or a woman, then each of the 128 outcomes should be equally likely, so we should assign probability $|A_j|/128$ to each event A_j

19. **GW** 2.33a

    Solution: $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. Since $\Pr(A \cap B) \geq 0$, we have $-\Pr(A \cap B) \leq 0$ and so $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$. Informally, the left-hand side counts all outcomes that are in at least one of A and B, and counts each one exactly once; the right-hand side counts the same outcomes, but those which are in both A and B are counted twice; so right-hand side is (potentially) bigger

20. **GW** 2.37

    Solution: There are 24 arrangements in all; these are listed in figure 3 of the figures page (with A standing for Alice, etc.). By inspection there are 8 arrangements in which neither couple (Bob and Catharine, Doug and Edna) are seated side-by side (the 8 with 0 marked in the center); so A_0 consists of 8 outcomes. Similarly there are exactly 8 outcomes in each of A_1, A_2. Since all arrangements are equally likely, we should assign probability $8/24 = 1/3$ to each A_i

21. **GW** 3.5

    a) **Solution**: Letting e stand for less than 4 hours, and l stand for more, there are 7 outcomes in which at least 1 arrives in less than 4 hours (with probabilities in parenthesis afterwards, computed by independence):

    $$eee(.75^3), eel(.75^2 \times .25), ele(.75^2 \times .25), lee(.75^2 \times .25), ell(.75 \times .25^2), lel(.75 \times .25^2), lle(.75 \times .25^2)$$

    Summing the probabilities of these disjoint events gives a probability of $.75^3 + 3 \times .75^2 \times .25 + 3 \times .75 \times .25^2 = .984375$

    a) **Solution**: There are 3 outcomes in which exactly 2 arrive in less than 4 hours (with probabilities in parenthesis afterwards, computed by independence):

    $$eel(.75^2 \times .25), ele(.75^2 \times .25), lee(.75^2 \times .25)$$

    Summing the probabilities of these disjoint events gives a probability of $3 \times .75^2 \times .25 = .421875$

22. **GW** 3.7

    Solution: $\Pr(A) = 2/6$, $\Pr(B) = 3/6$, $\Pr(A \cap B) = 1/6 = 2/6 \times 3/6 = \Pr(A) \times \Pr(B)$ so Yes, they are independent

23. **GW** 3.10

    We can do all four parts by listing the specific outcomes in the event described, computing the probabilities of each individual outcome by independence, and then summing. In what follows, the outcome rwr (for example) means that I guess right for the first question, wrong for the second and right for the third. Since I have a $1/5 = .2$ probability of guessing right and so a .8 probability of guessing wrong for each one, and the guesses are independent, the probability of this particular outcome is $.2 \times .8 \times .2 = .2^2 \times .8$.

    a) **Solution**: $\Pr(rrr) = .2^3 = .08$

    b) **Solution**: $\Pr(www) = .8^3 = .512$

c) **Solution**: $\Pr(rww \text{ or } wrw \text{ or } wwr) = 3 \times .2 \times .8^2 = .384$

d) **Solution**: $\Pr(rrw \text{ or } rww \text{ or } wrr) = 3 \times .2^2 \times .8 = .096$

a) **Solution**: They do, as they should, since one of the four disjoint events (get none right, one right, two right, three right) must occur

24. **GW** 3.12

Solution: Looking at all 24 outcomes (figure 3 of the figures page) and counting, we find that in 12 of them, Bob and Catherine are sitting together, so $\Pr(T) = 1/2$. Similarly, $\Pr(U) = 1/2$. There are 4 outcomes in which both Bob and Catherine are sitting together, as well as Bob and Alice, so $\Pr(T \cap U) = 1/6$. This is not the same as $\Pr(T)\Pr(U)$, so these events are *not* independent.