Math 30530: Introduction to Probability, Fall 2012

Midterm Exam I, October 10

Solutions

1. Every Sunday, I watch 4 NFL games. On Sunday morning I try to predict the winners of each of these games. I reckon I do this well; I assess that my probability of correctly predicting the winner of each game is 2/3 (all predictions independent).

   (a) (4 pts.) On a given Sunday, what is the probability that I correctly predict the winners of at least 3 games?

   Solution: Let X be the number of winners I successfully predict; $X \sim \text{Binomial}(4, 2/3)$, so $\Pr(X \geq 3) = \binom{4}{3}(2/3)^3(1/3) + (2/3)^4 = 48/81 \approx .59$.

   (b) (3 pts.) Over a 12 week span, what is the expected number of weeks on which I correctly predict the winners of at least 3 games?

   Solution: Let Y be the number weeks on which I correctly predict the winners of at least 3 games; $Y \sim \text{Binomial}(12, 48/81)$, so $E(Y) = 12 \times (48/81) = 192/27 \approx 7.1$.

   (c) (3 pts.) Write down (but don't bother evaluating) an expression that equals the probability that on at least 7 of the 12 weeks, I correctly predict the winners of at least 3 games.

   Solution: $\Pr(Y \geq 7) = \sum_{k=7}^{12} \binom{12}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{12-k}$.

2. (2 pts. each) For the True or False questions, if you answer "false", say what needs to be added to make the statement true.

   (a) **True or False**: One of the basic rules of probability is that the probability of the union of a collection of events is the sum of the probabilities of the events.

   Solution: False. The events need to be mutually exclusive for this to be true.

   (b) **True or False**: For events A and B, $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

   Solution: False. The events need to be independent for this to be true.

   (c) **True or False**: Mutually exclusive events are sometimes independent, and sometimes not.

   Solution: True. If $A = \emptyset$ and B is anything, then $\Pr(A \cap B) = 0$ and $\Pr(A)\Pr(B) = 0$, so A and B are mutually exclusive and independent; but if mutually exclusive A and B satisfy $\Pr(A), \Pr(B) > 0$, then $\Pr(A \cap B) = 0 \neq \Pr(A)\Pr(B)$, so there are *not* independent.

   (d) **Fill in the blank**: The negative binomial random variable with parameters r and p has expectation

   $$\underline{r/p}.$$

(e) **Fill in the blank**: If A, B and C are three events, then

$$\Pr(A \cup B \cup C) = \underline{\Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) + \Pr(A \cap C) + \Pr(B \cap C) + \Pr(A \cap B \cap C)}.$$

3. In my hand I have five playing cards, two of which have face value 2 and three of which have face value 3. I choose two cards, one after the other, *with replacement*, and note the face values. Jack is interested in X, the larger of the two values, and Jill is interested in Y, the difference between the larger and the smaller values.

(a) (7 pts.) Calculate the joint mass function of X and Y (a table of values is ok).

Solution: One possible pair of values is $X = 2, Y = 0$, which happens only if I pick a 2 both times, probability $4/25$. Another possibility is $X = 2, Y = 1$, which can never happen, so probability 0. Another possibility is $X = 3, Y = 0$, which happens only if I pick a 3 both times, probability $9/25$. The final possibility is is $X = 3, Y = 1$, which happens only if the two cards I pick have different face values, probability $12/25$.

(b) (3 pts.) Given that $Y = 0$, what is the probability that $X = 2$?

Solution: $\Pr(X = 2|Y = 0) = \frac{\Pr(X=2,Y=0)}{\Pr(Y=0)} = \frac{4/25}{4/25+9/25} = 4/13$.

4. When I go away for a week, I ask my neighbor to regularly water my plants. I know from experience that she is 90% likely to actually do this. If my plants are regularly watered, they stay healthy with probability .9. If my plants are not regularly watered, they stay healthy with probability .2.

(a) (5 pts.) What is the probability that my plants will be healthy when I return from my trip?

Solution: Let H be the event that my plants are healthy, and W the event that my neighbor came by to water them. Then

$$\Pr(H) = \Pr(H|W)\Pr(W) + \Pr(H|W^c)\Pr(W^c) = (.9)(.9) + (.2)(.1) = .83.$$

(b) (5 pts.) I come home from my trip, and find that my plants are not healthy. What is the probability that my neighbor failed to come by to regularly water them?

Solution: By definition of conditional probability,

$$\Pr(W^c|H^c) = \frac{\Pr(W^c \cap H^c)}{\Pr(H^c)} = \frac{\Pr(H^c|W^c)\Pr(W^c)}{\Pr(H^c)} = \frac{(.8)(.1)}{.17} \approx .48.$$

5. (a) (7 pts.) In how many ways can the cards in an ordinary deck of 52 cards be arranged in such a way that the four aces occur consecutively, and the four kings occur consecutively as well?

Solution: We can imagine gluing the four aces together, and gluing the four king together; the deck then has 46 cards, and there are 46! ways to arrange these cards. But, there are 4! ways in which the 4 aces can be glued together, and the same for the 4 kings; so the total count is 49!4!4!.

(b) (3 pts.) An ordinary deck of 52 cards is thoroughly shuffled. What is the probability that the four aces will occur consecutively, and the four kings will occur consecutively as well? (**Hint**: use your result from the previous part!)

Solution: There are 52! ways of arranging the 52 cards, and 46!4!4! of those ways have the four aces occurring consecutively, and the four kings occurring consecutively as well; so the probability is $(46!4!4!)/52! \approx 4 \times 10^{-8}$.

6. I have four envelopes. Two of them have cards with the number "2" written on them, one has the number "0" and one has the number "8". You pick an envelope at random. Let X be the number written on the card inside that envelope.

(a) (3 pts.) Compute $E(X)$.

Solution: X is 0 with probability 1/4, 2 with probability 1/2, and 6 with probability 1/4, so $E(X) = 0(1/4) + 2(1/2) + 8(1/4) = 3$.

(b) (4 pts.) In the *square game*, you win, in dollars, the square of the number shown on the envelope. Calculate your expected winning in the square game.

Solution: The amount you win is X^2. $E(X^2) = 0(1/4) + 2^2(1/2) + 8^2(1/4) = 18$.

(c) (3 pts.) Compute $\mathrm{Var}(X)$.

Solution: $\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 18 - (3)^2 = 9$.