Math 30530: Introduction to Probability, Fall 2011

Solutions to selected homeworks

October 2, 2011

1 Chapter 1

- **Theoretical exercise 2**: If the first experiment results in outcome 1, there are n_1 possible outcomes for the second stage. If it results in outcome 2, there are n_2 possibilities. So if the first experiment results in either outcome 1 or outcome 2, then there are $n_1 + n_2$ possible outcomes for the second stage, and so n_1+n_2 possible outcomes for the whole experiment. More generally, with m possible outcomes for stage one, the number of possible outcomes for the whole experiment is $n_1+n_2+\ldots+n_m = \sum_{i=1}^{m} n_i$.

2 Chapter 3

- **Problem 66**: The simplest approach here is to simply list all of the configurations of open/closed relays that result in a closed path from A to B, compute the probability of each one (using independence) and then add up all the individual probabilities. For example, for A, the following are the configurations that create a closed path from A to B (I'll list the possibilities by putting "O" after a relay f it is open, so not working, and "C" after one that is closed, so working): 1C2C3O4O5C, 1C2C3C4O5C, 1C2C3O4C5C, 1C2C3C4C5C, 1C2O3C4C5C, 1O2C3C4C5C, 1O2O3C4C5C. It requires a little care to make sure a) that all possibilities are listed, and b) that none are listed twice. The probability of the first of these is, by the independence of the relays, $p_1p_2(1-p_3)(1-p_4)p_5$. One writes down a similar expression for each of the other six possibilities, and adds them all up to get the total probability that there is a closed path from A to B. The strategy for part b) is identical.

3 Chapter 4

- **Problem 20**: Here are the possibilities for this strategy: W, LWW, LWL, LLW, LLL. The probability of the first is $18/38$, and results in a net profit of \$1. The probability of the second is $(20/38)(18/38)(18/38)$, and results in a net profit of \$1. The probability of the third is $(20/38)(18/38)(20/38)$, and results in a net loss of \$1. The probability of the fourth is $(20/38)(20/38)(18/38)$, and results in a net loss of \$1. The probability of the fifth is $(20/38)(20/38)(20/38)$, and results in a net loss of \$3. So the mass function of X, the number of dollars you win, is $p(1) = 18/38 + (20/38)(18/38)(18/38) = .59...$; $p(-1) = (20/38)(18/38)(20/38) + (20/38)(20/38)(18/38) = .26...$; $p(-3) = (20/38)(20/38)(20/38) = .15....$
  - a) $P(X > 0) = .59...$

- b) It seems promising, since you are more likely to make a loss than a profit. *But*, when you make a profit, it's only \$1, while there are some scenarios where you make a loss of \$3. So it's possible that in the long-run, the 15% of the time when you make a \$3 loss will hurt you. It part c), we'll see that it does, and so this is *not* a winning strategy.
- c) $E(X) \approx .59 - .26 - 3 \times .15 = -.12$. So in the long run, although you expect to win more than you lose, you also expect to lose on average 12 cents per trial of this strategy.

- **Theoretical exercise 4**: This one is easier if we write out the expectation in full (rather than using summation notation).

$$E(N) = 0P(N = 0) + 1P(N = 1) + 2P(N = 2) + 3P(N = 3) + \ldots.$$

This is by definition. Since $0P(N = 0) = 0$, we can replace this with

$$E(N) = 1P(N = 1) + 2P(N = 2) + 3P(N = 3) + \ldots. \tag{1}$$

Now

$$P(N \geq 1) = P(N = 1) + P(N = 2) + P(N = 3) + \ldots.$$

So if we take away $P(N \geq 1)$ from the right hand side of (1), we are left with

$$1P(N = 2) + 2P(N = 3) + \ldots.$$

We continue. We have

$$P(N \geq 2) = P(N = 2) + P(N = 3) + \ldots.$$

So if we now take away $P(N \geq 1)$ from the right hand side of (1), we are left with

$$1P(N = 3) + \ldots.$$

If we keep going, taking away $P(N \geq 3)$, then $P(N \geq 4)$, and continue this process indefinitely, that we end up taking away *all* of the right hand side of (1), and nothing else. For example, when we come to take away $P(N \geq 17)$, what we have left begins $1P(N = 17) + 2P(N = 18) + \ldots$, and so when we have finished takin away $P(N \geq 17)$, there will be no $P(N = 17)$ left, and none will be needed in the future ($P(N = 17)$ doesn't appear in the full expression for $P(N \geq 18)$, or $P(N \geq 19)$, etc). So we conclude that

$$E(N) = P(N \geq 1) + P(N \geq 2) + P(N \geq 3) + \ldots.$$