Math 30530 — Introduction to Probability

Fall 2009 final exam

December 17, 2009, 1.45pm-3.45pm

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This examination contains 7 problems on 8 pages. It is open-book, open-notes. You may use a calculator. **Show all your work** on the paper provided. The honor code is in effect for this examination.

Scores

Question	Score	Out of
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

GOOD LUCK !!!

1. My wife participates in a football pool at work with seven other people. Each week during the Colts' 16 week regular season, each person pays in \$2 (for a total of \$16), and the \$16 dollars goes to one person in a competition based on the final score of that week's Colts game. Assume that competitions from week to week are independent, and that each week each person has a 1/8 chance of winning.

   (a) Let X be the number of times my wife wins during the 16 week season. What is the expectation and variance of X?

$$X = \text{Binomial}\left(n = 16, p = \tfrac{1}{8}\right)$$

$$E(X) = np = 2$$

$$Var(X) = np(1-p) = 2 \times \tfrac{7}{8} = \tfrac{7}{4}$$

   (b) What is $P(X \geq 2)$? (Since she puts in a total of \$32, this is the probability that she at least breaks even in the football pool.)

$$P(X=0) = \binom{16}{0}\left(\tfrac{1}{8}\right)^0\left(\tfrac{7}{8}\right)^{16} = .1181$$

$$P(X=1) = \binom{16}{1}\left(\tfrac{1}{8}\right)^1\left(\tfrac{7}{8}\right)^{15} = .2699$$

$$\text{So } P(X \geq 2) = 1 - P(X=0) - P(X=1) = .6121$$

   (c) Let Y be the number of people in the pool who at least break even over the whole 16 weeks. Compute $E(Y)$.

~~Y = Bin~~ let $Y_i = \begin{cases} 1 & \text{if person } i \text{ breaks even} \\ 0 & \text{owise} \end{cases}$

$E(Y_i) = .6121$ (by part b)

$Y = Y_1 + \dots + Y_{16}$

$E(Y) = E(Y_1) + \dots + E(Y_{16}) = 16 \times .6121$

$= 9.79$

2. I have nine balls, numbered 1 through 9. Balls 1 through 3 are red, and the remaining six balls are green. I also have three boxes labeled A, B and C. I put the balls into the three boxes so that each box gets exactly 3 balls. (The order within each box doesn't matter.)

   (a) In how many ways can I do this?

9! ways if order within boxes mattered
(order the balls; put first three in box A, etc.)
To eliminate order, divide by 3! for each box

So $\frac{9!}{(3!)^3} = 1680$ ways

   (b) What is the probability that all three red balls go into the same box?

\# ways for them to go into box A: $\frac{3! \times 6!}{(3!)^3} = 20$

So \# ways for them to go into same box $= 60$

$P(\text{same box}) = \frac{60}{1680} = .0357\ldots$

   (c) What is the probability that there ends up being one red ball in each of the three boxes?.

\# ways to first distribute 3 red balls among the boxes $= 3! = 6$

\# ways to distribute remaining green balls
$= \frac{6!}{(2!)^3} = 90$

So 540 ways

$P(\text{reds in 3 different boxes}) = \frac{540}{1680} = .32\ldots$

3. Ray Rice of the Baltimore Ravens averages 4 yards per carry with a standard deviation of 1.2.

   (a) Use an appropriate inequality to put a lower bound on the probability that in three successive carries his net gain is between 8 and 16 yards (assume that different carries are independent of each other).

$X = X_1 + X_2 + X_3$ = rv with mean $3 \times 4 = 12$

variance $3 \times (1.2)^2 = 4.32$

X_i = yards in i^{th} carry

Tchebychev: $P(8 \le X \le 16)$

$= P(|X - 12| \le 4)$

$= P(|X - \text{mean}| \le 1.92\ldots \text{ std dev})$

$\ge 1 - \frac{1}{(1.92\ldots)^2} = .73$

   (b) Given the extra information that Rice's yards per carry is normally distributed, compute the exact probability that in three successive carries his net gain is between 8 and 16 yards.

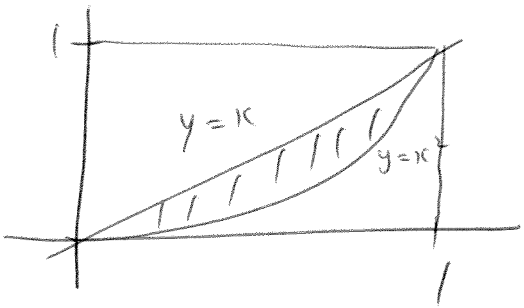
$X = \text{Normal}(\mu = 12, \sigma^2 = 4.32)$

$P(8 \le X \le 16) = P\left(\frac{-4}{\sqrt{4.32}} \le Z \le \frac{4}{\sqrt{4.32}}\right)$

$= P(-1.92\ldots \le Z \le 1.92\ldots)$

$= .9452$

4. (a) X and Y are two continuous random variables which have a joint density $f(x, y)$ that is non-zero only on the square $[0, 1] \times [0, 1]$. Write down an integral whose value is the probability that $X^2 \le Y \le X$.



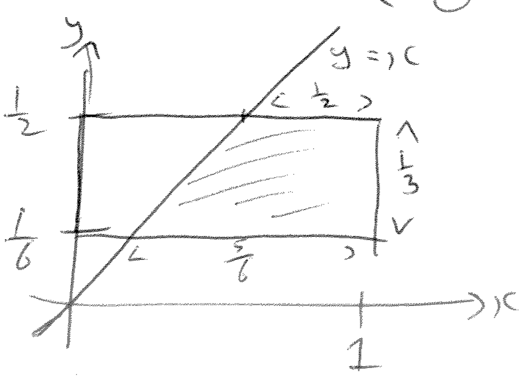
$$P(X^2 \le Y \le X) = \int_{x=0}^{1} \int_{y=x^2}^{x} f(x,y)\, dy\, dx$$

(b) An introverted professor X rarely turns her face away from the blackboard. The moment when she first faces her students is equally likely to occur at any point during her hour-long lecture. X's student, Y, is very busy. He's always at least 10 minutes late, though he always manages to get to class (at a completely random moment) before the lecture is halfway through. How likely is it that when X faces the class for the first time, she'll see Y eagerly taking notes?

$X = \text{uniform}(0,1)$
$Y = \text{uniform}(\frac{1}{6}, \frac{1}{2})$ } units = hours

X, Y independent, so joint density of (X, Y) is

$$f(x,y) = \begin{cases} 3 & \text{on } \text{rectangle } [0,1] \times [\frac{1}{6}, \frac{1}{2}] \\ 0 & \text{elsewhere} \end{cases}$$



We want $P(Y \le X)$

$$= \iint f(x,y)\, dA = 3 \times \text{Area}(\ ) $$

$$= 3 \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{3}$$

average of $\frac{1}{2}, \frac{5}{6}$

5. (a) I toss a coin, and let X be the number of heads I get and Y the number of tails (so both X and Y take values either 0 or 1).

i. Compute the covariance of X and Y.

$X = \begin{cases} 1 \text{ w prob } \frac{1}{2} \\ 0 \text{ w prob } \frac{1}{2} \end{cases}$, same for Y, so $E(X) = E(Y) = \frac{1}{2}$

$XY = 0$ always $\left(\begin{matrix} \text{if } X=1, Y=0 \\ \text{if } X=0, Y=1 \end{matrix} \right)$, so $E(XY) = 0$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0 - \frac{1}{4} = -\frac{1}{4}$

ii. Are X and Y independent? Briefly justify.

No; Cov $\neq 0$, and also, knowing value of X tells you value of Y.

(b) For general random variables X and Y, show that $\mathrm{Cov}(X+Y, X-Y) = 0$ if and only if $\mathrm{Var}(X) = \mathrm{Var}(Y)$

$$\begin{aligned}
\mathrm{Cov}(X+Y, X-Y) = 0 &\iff E((X+Y)(X-Y)) = E(X+Y)E(X-Y) \\
&\iff E(X^2 - Y^2) = (E(X)+E(Y))(E(X)-E(Y)) \\
&\iff E(X^2) - E(Y^2) = (E(X))^2 - (E(Y))^2 \\
&\iff E(X^2) - (E(X))^2 = E(Y^2) - (E(Y))^2 \\
&\iff \mathrm{Var}(X) = \mathrm{Var}(Y)
\end{aligned}$$

6. (a) I want use an exponential random variable to model the time I have to wait (in minutes) until the first person asks a question during today's exam. Describe in words what I should choose for the parameter λ.

λ = average # questions asked per minute

(b) Calls arrive at a telephone exchange at a rate of 10 per minute. Using an appropriate random variable to model the number of calls arriving at the exchange, compute the probability that more than two calls arrive in a space of 6 seconds.

~~Use Poisson ($\lambda = 10$~~) Take time unit to be 1 unit = 6 seconds. Average # calls arriving per unit is 1 (10/min = 1/6 secs)

So use Poisson ($\lambda = 1$)

$P(\geq 2 \text{ calls}) = 1 - P(0 \text{ or } 1 \text{ call}) = 1 - e^{-1} - \frac{1}{1!}e^{-1}$

$= .264\ldots$

(c) If X is an exponential random variable with $P(X \geq 2) = P(X < 2)$, compute its parameter λ.

$P(X \geq 2) = \int_2^\infty \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_2^\infty = e^{-2\lambda}$

We want this to be $\frac{1}{2}$ (Since $P(X \geq 2) + P(X < 2) = 1$, and these are equal)

So want $e^{-2\lambda} = \frac{1}{2}$

$e^{2\lambda} = 2$

$2\lambda = \ln 2$

$\lambda = \frac{\ln 2}{2}$

7. At a given moment, the velocity v of a particular particle (measured in meters per second) is a random variable with the following density:

$$f(x) = \begin{cases} \frac{1}{10}(4x+1) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the expectation of the velocity.

$$E(v) = \int_0^2 \frac{x}{10}(4x+1)\,dx = \left[\frac{4x^3}{30} + \frac{x^2}{20}\right]_0^2$$

$$= \frac{4 \times 8}{30} - \frac{4}{20} = \frac{13}{15}$$

(b) Calculate the probability that the particle has a velocity greater than the mean velocity.

$$P(v > E(v)) = P\left(v \geq \frac{13}{15}\right) =$$

$$\int_{\frac{13}{15}}^{2} \frac{1}{10}(4x+1)\,dx = \left[\frac{4x^2}{20} + \frac{x}{10}\right]_{\frac{13}{15}}^{2} = \frac{4 \times 2^2}{20} + \frac{2}{5} - \frac{4\left(\frac{13}{15}\right)^2}{20} - \frac{15}{10}$$

$$= \text{Something}$$

(c) Compute the distribution function of the particle's kinetic energy $K = mv^2/2$. (Here m, the mass of the particle, is a certain constant.)

K takes values between 0 and $\frac{1}{2}m2^2 = 2m$

So $P(K \leq a) = \begin{cases} 0 & \text{if } a \leq 0 \\ 1 & \text{if } a \geq 2m \end{cases}$

For $0 < a < 2m$,

$$P(K \leq a) = P\left(\tfrac{1}{2}mv^2 \leq a\right) = P\left(v \leq \sqrt{\tfrac{2a}{m}}\right)$$

$$= \int_0^{\sqrt{\frac{2a}{m}}} \frac{1}{10}(4x+1)\,dx = \left[\frac{4x^2}{20} + \frac{x}{10}\right]_0^{\sqrt{\frac{2a}{m}}}$$

$$= 4\left(\frac{2a}{m}\right)/20 + \sqrt{\frac{2a}{m}}\,/10 .$$