MATH 30530 **Introduction to Probability** **Fall 2011**

Practice problems for second midterm

1. Assume that Θ is uniformly distributed on the interval $(-\pi/2, \pi/2)$. Let $Y = \sin\Theta$. (Y is the y-coordinate of a randomly chosen point on the unit circle.)

A. Find the distribution function of Θ.

Solution: $F_\Theta(x) = 0$ if $x \leq -\pi/2$, and $F_\Theta(x) = 1$ if $x \geq \pi/2$. For $-\pi/2 < x < \pi/2$, we have

$$F_\Theta(x) = \int_{-\pi/2}^{x} \frac{1}{\pi}\, dt = \frac{x + \pi/2}{\pi}.$$

B. Find the distribution function of Y.

Solution: The range of possible values for $\sin\theta$ as θ goes from $-\pi/2$ to $\pi/2$ is $[-1, 1]$. So $F_Y(y) = 0$ if $y < -1$, and $F_Y(y) = 1$ if $x > 1$. For $-1 \leq x \leq 1$, we have

$$F_Y(y) = P(Y \leq y) = P(\sin\Theta \leq y) = P(\Theta \leq \sin^{-1} y) = F_\Theta(\sin^{-1} y) = \frac{\sin^{-1} y + \pi/2}{\pi}.$$

C. Find the density of Y.

Solution: We have $f_Y(y) = 0$ if $y \notin [-1, 1]$. For $-1 \leq y \leq 1$, we have

$$f_Y(y) = \frac{d}{dy}\left(\frac{\sin^{-1} y + \pi/2}{\pi}\right) = \frac{1}{\pi\sqrt{1 - y^2}}.$$

2. Adam rolls a fair die until he gets a 6. Independently, Andrew rolls a fair die until he gets an odd number.

A. On average, how many times does Adam roll his die?

Solution: If D is the number of rolls Adam must make, then D is a geometric random variable with $p = 1/6$, so $E(D) = 6$.

B. On average, how many times Andrew roll his die?

Solution: If N is the number of rolls Andrew must make, then N is a geometric random variable with $p = 3/6 = 1/2$, so $E(D) = 2$.

C. What is the probability that Adam rolls his die more times than Andrew rolls his?

Solution: We want $P(D > N)$. If $N = 1$, this is $P(D > 1) = (5/6)$. If $N = 2$, this is $P(D > 2) = (5/6)^2$. In general, if $N = k$, this is $P(D > k) = (5/6)^k$ (since $D > k$ means exactly failure on the first k rolls). By the law of total probability,

$$P(D > N) = \sum_{k\geq 1} P(D > N|N = k)P(N = k) = \sum_{k\geq 1}(5/6)^k(1/2)^k = \frac{5/12}{1-(5/12)} = 5/7.$$

3. (20 points) While imprisoned in the Chateau d'If, Edmond Dantès and the Abbé Faria play Monopoly each evening. However, they have lost a die they need for the game. Dantès has made a replacement. One test of the new die will be to roll it 2880 times. If it comes up 6 between 460 and 500 times, they will conclude that the die comes up 6 about one sixth of the time. If the die is fair, what is the probability that the test leads to an incorrect conclusion?

Solution: If the dice is fair, then X, the number of 6's that occur in 2880 rolls, is a Binomial random variable with $p = 1/6$ and $n = 2880$, so $E(X) = 480$ and $\mathrm{Var}(X) = 400$. By the DeMoivre-Laplace theorem it follows that $(X - 480)/20$ is approximately a standard normal. We want $P(X < 460) + P(X > 500)$. We have

$$\begin{aligned} P(X < 460) + P(X > 500) &= P((X - 480)/20 < -1) + P((X - 480)/20 > 1) \\ &= P(Z < -1) + P(Z > 1) = .3174. \end{aligned}$$

4. (20 points) The time it takes for a calculus student to answer all the questions on a certain exam is an exponentially distributed random variable with mean 1 hour and 15 minutes. If 10 students are taking the exam, what is the probability that at least one of them completes it in less than 1 hour?

Solution: Let X be the time it takes a particular student to finish. Since X is exponential with mean 1.25 (time measured in hours), it follows that $\lambda = .8$ (remember that $E(X) = 1/\lambda$). So the probability that the particular student finishes in less than 1 hour is

$$P(X < 1) = \int_0^1 .8e^{-.8x}\,dx = [-e^{-.8x}]_{x=0}^{x=1} = 1 - e^{-.8} = .55....$$

Assuming that the students' finishing times are independent, the probability that none of them finishes in less than one hour is

$$(1-(1-e^{-.8}))^{10} = e^{-8} = 3.35... \times 10^{-4}.$$

So the probability that at least one of them finishes in less than one hour is

$$1 - e^{-8} = .99966....$$

5. The joint density of a pair of random variables X, Y is

$$f(x,y) = \begin{cases} Cxe^{-4y} & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

A. Find C.

Solution: We compute

$$\int_{x=0}^{4}\int_{y=0}^{\infty} Cxe^{-4y}\,dydx = C\int_{x=0}^{4} x\,dx \int_{y=0}^{\infty} e^{-4y}\,dy = C[x^2/2]_0^4[-e^{-4y}/4]_0^{\infty} = 2C.$$

Since we must have the double integral be 1, we must have $C = 1/2$.

B. Are X and Y independent? Give a brief explanation.

Solution: Because the density function is non-zero on a rectangle, and on that rectangle $f(x,y)$ factors into terms involving x only and terms involving y only, the two random variable X and Y are independent.

C. Find the marginal density of X.

Solution: $f_X(x) = 0$ for $x < 0$ and for $x > 4$. For $0 \le x \le 4$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} \frac{xe^{-4y}}{2}\,dy = \frac{x}{2}[-e^{-4y}/4]_0^{\infty} = \frac{x}{8}.$$

D. Write down (but don't evaluate) an integral whose value is the probability that $X + Y \ge 4$.

Solution:

$$\int_0^4 \int_{4-x}^{\infty} \frac{xe^{-4y}}{2}\,dydx = .$$

6. Let X and Y be independent random variables, each uniformly distributed on $[0,1]$. Find the probability that the quadratic equation

$$t^2 - Xt + Y = 0$$

has real roots.

Solution: By the quadratic formula, the condition that ensures real roots is $X^2 - 4Y \geq 0$, or $Y \leq X^2/4$. The probability that this relation holds is just the area under the curve $y = x^2/4$ between $x = 0$ and $x = 1$. This is

$$\int_0^1 \frac{x^2}{4}\, dx = 1/12.$$

7. I arrive at the bus stop by the bookstore at noon. I know that the number of hours that will pass before the bus to Midway arrives is a random variable X that is uniformly distributed on $[0, 1/2]$. The National Weather service tells me that it will begin raining sometime after noon, and that the number of hours that will pass before the rain starts is an exponentially distributed random variable Y with $\lambda = 2$.

A. Write down the joint density function $f(x, y)$ of X and Y.

Solution: Unless both $0 \leq x \leq 1/2$ and $0 \leq y \leq \infty$, the joint density $f(x, y)$ is 0. For $0 \leq x \leq 1/2$ and $0 \leq y < \infty$, we have

$$f(x, y) = 2 \times 2e^{-2y} = 4e^{-2y}.$$

B. Compute the probability that it will begin to rain before the bus arrives.

Solution: We want $P(Y < X)$. The region of the plane in which the joint density is non-zero, and $y < x$, is the triangle with vertices $(0, 0)$, $(1/2, 0)$ and $(1/2, 1/2)$. So

$$P(Y < X) = \int_{x=0}^{1/2} \int_{y=0}^{x} 4e^{-2y}\, dydx = \int_0^{1/2} (2 - 2e^{-2x})\, dx = e^{-1}.$$