MATH 30530 **Introduction to Probability** **November 16, 2011**

Midterm Examination II Solutions

1. The joint probability mass function $p(i, j)$ of the discrete random variables X and Y is given in the table below.

		Y		
		1	2	3
X	1	2/27	4/27	2/27
	2	1/27	7/27	4/27
	3	1/27	5/27	1/27

A. (5 points) Find the marginal probability mass function of X.

Solution: Summing across rows, we find that $p_X(1) = 8/27$, $p_X(2) = 12/27$ and $p_X(3) = 7/27$. For all other values of x, $p_X(x) = 0$. (Note that this last piece of information is strictly speaking necessary to fully specify p_X.)

B. (5 points) Find the marginal probability mass function of Y.

Solution: Summing down columns, we find that $p_Y(1) = 4/27$, $p_Y(2) = 16/27$ and $p_Y(3) = 7/27$. For all other values of y, $p_Y(y) = 0$.

C. (10 points) Find $P\{X < Y\}$.

Solution: The pairs corresponding to $X < Y$ are $X = 1, Y = 2$, $X = 1, Y = 3$ and $X = 2, Y = 3$, so $P(X < Y) = (4/27) + (2/27) + (4/27) = 10/27$.

2. In order to conduct her latest experiment, Professor X needs to find 8 students with blood type O+. She knows that an individual has a 40% chance of having this blood type. In order to find her 8 subjects, she tests the blood of randomly chosen students, one after another.

A. (7 points) What is the probability that she has to test exactly 20 students to find 8 with the right blood type?

Solution: If X is the number of students needed to be tested, then X is a negative binomial random variable with $r = 8$ and $p = .4$, so $P(X = 20) = \binom{19}{7}(.4)^8(.6)^{12}$.

B. (7 points) If each test she conducts costs \$75, what the expected total cost of Professor X's effort to find 8 students with with the right blood type?

Solution: We want $E(75X) = 75E(X)$. Since the expectation of a negative binomial with parameters r and p is r/p, we have $E(X) = 8/.4 = 20$ and $E(75X) = 1500$.

C. (6 points) After finding her 8 students, Professor X discovers that one of the 8 is hemophiliac and cannot take part in the experiment. What is the expected number of students that she has to test to find a replacement subject?

Solution: If Y is the number of new students needed to be tested, then Y is a geometric random variable with $p = .4$, so $E(Y) = 1/.4 = 2.5$. Note that it is **not** correct to round this up to 3 or down to 2, on the grounds that Professor X can only test a whole number of students. The expectation of a random variable has a precise meaning, and it need not be one of the values that the random variable takes on.

3. I'm taking part in the All-Ireland hay-tossing championship next week. The distance I can throw a bale of hay (in yards, from where I'm standing) is a random variable with density function

$$f(x) = \begin{cases} \frac{c}{x^4} & \text{for } x \geq 2 \\ 0 & \text{for } x < 2 \end{cases}$$

A. (6 points) What is c?

Solution: Since $\int_2^\infty c/x^4\,dx = c/24$, we must have $c = 24$.

B. (8 points) What is the probability that I throw the bale between 2 and 3 yards?

Solution: This is $\int_2^3 24/x^4\,dx = 19/27$.

C. (6 points) The prize a contestant receives is $100x^2 + 100x$ euro if he tosses the bale x yards. What is my expected prize?

Solution: We need to calculate not the expectation of distance, but the expectation of a function g of the distance, where the function is $g(x) = 100x^2 + 100x$, so we use

$$\int_{-\infty}^{\infty} g(x)f(x)\,dx = \int_2^\infty 24(100x^2 + 100x)/x^4\,dx = 1500.$$

Note that it **not** correct to calculate the expected distance (which turns out to be 3) and then apply g to this. This is because $E(g(X)) \neq g(E(X))$ in general (and in particular, $E(X^2 + X) \neq E(X)^2 + E(X)$).

4. One hour after a rod is taken out of an annealing chamber, its temperature is normally distributed with mean 136 and variance 64. To be tempered, the rod should have a temperature below 124.

A. (10 points) Calculate the probability that a rod is ready for tempering one hour after being removed from the annealing chamber.

Solution: Let T be the temperature after an hour. We want

$$P(T < 124) = P((T - 136)/8 < -1.5) = P(Z < -1.5) = .0668.$$

B. (10 points) Suppose 400 rods come out of the annealing chamber at once. Assuming that the temperatures of the rods after one hour are independent, use the DeMoivre-Laplace Theorem to estimate the probability that at least 35 of the rods are ready for tempering after one hour.

Solution: Let X be the number of rods ready for tempering after an hour. Since X is binomial with $n = 400$ and $p = .0668$, it's mean and standard deviation are 26.72 and 4.9935, respectively, we use DeMoivre-Laplace to say

$$P(X \geq 35) = P((X - 26.72)/4.9953 \geq 1.66) = P(Z > 1.66) = .0485.$$

Note that I'm not using the continuity correction. If I was, I would calculate instead $P(X \geq 34.5)$.

5. Let the random variables X and Y have joint density

$$f(x,y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } x > 0 \text{ and } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

A. (6 points) Find the marginal densities f_X and f_Y.

Solution: If general, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$. Applying here, we get

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that to fully specify the marginal densities, we need to give their values for **all** possible inputs (including those intervals where the densities are 0).

B. (6 points) Are X and Y independent? Explain.

Solution: Because $f_X(x)f_Y(y) = f(x,y$ for all x, y, they are independent.

C. (8 points) Find $P\{X < Y\}$.

Solution: After drawing a diagram, we see that

$$P(X < Y) = \int 0^2 \int 0^y e^{-x}/2\,dxdy = \int 0^2 \int_x^2 e^{-x}/2\,dydx = (1 + e^{-2})/2.$$

(The first of these is much easier to calculate than the second.)