MATH 30530, Fall 2009 – Homework 9 Solutions

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1. Let G be the distribution function of Y; for $-8 < y < 8$,

$$G(y) = P(Y \le y) = P(X^3 \le y) = P(X \le \sqrt[3]{y}) = \int_{-2}^{\sqrt[3]{y}} \frac{1}{4}\,dx = \frac{1}{4}\sqrt[3]{y} + \frac{1}{2}.$$

Therefore,

$$G(y) = \begin{cases} 0 & y < -8 \\ \dfrac{1}{4}\sqrt[3]{y} + \dfrac{1}{2} & -8 \le y < 8 \\ 1 & y \ge 8. \end{cases}$$

This gives

$$g(y) = G'(y) = \begin{cases} \dfrac{1}{12} y^{-2/3} & -8 < y < 8 \\ 0 & \text{otherwise.} \end{cases}$$

Let H be the distribution function of Z; for $0 \le z < 16$,

$$H(z) = P(X^4 \le z) = P(-\sqrt[4]{z} \le x \le \sqrt[4]{z}) = \int_{-\sqrt[4]{z}}^{\sqrt[4]{z}} \frac{1}{4}\,dx = \frac{1}{2}\sqrt[4]{z}.$$

Thus

$$H(z) = \begin{cases} 0 & z < 0 \\ \dfrac{1}{2}\sqrt[4]{z} & 0 \le z < 16 \\ 1 & z \ge 16. \end{cases}$$

This gives

$$h(z) = H'(z) = \begin{cases} \dfrac{1}{8} z^{-3/4} & 0 < z < 16 \\ 0 & \text{otherwise.} \end{cases}$$

2. Let G be the probability distribution function of Y and g be its probability density function. For $t > 0$,

$$G(t) = P\big(e^X \le t\big) = P(X \le \ln t) = F(\ln t).$$

For $t \le 0$, $G(t) = 0$. Therefore,

$$g(t) = G'(t) = \begin{cases} \dfrac{1}{t} f(\ln t) & t > 0 \\ 0 & t \le 0. \end{cases}$$

5. Let G and g be the probability distribution and the probability density functions of Y, respectively. Then

$$G(y) = P(Y \le y) = P\big(\sqrt[3]{X^2} \le y\big) = P(X \le y\sqrt{y})$$
$$= \int_0^{y\sqrt{y}} \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda y\sqrt{y}}, \quad y \in [0,\infty).$$

So

$$g(y) = G'(y) = \frac{3\lambda}{2}\sqrt{y}e^{-\lambda y\sqrt{y}}, \quad y \ge 0;$$

0, otherwise.

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5. $E(X) = \displaystyle\int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = 0$, because the integrand is an odd function.

9. The expected value of the length of the other side is given by

$$E\big(\sqrt{81-X^2}\big) = \int_2^4 \sqrt{81-x^2}\cdot\frac{x}{6}\,dx.$$

Letting $u = 81 - x^2$, we get $du = -2x\,dx$ and

$$E\big(\sqrt{81-X^2}\big) = \frac{1}{12}\int_{65}^{77} \sqrt{u}\,du \approx 8.4.$$

11. Note that

$$E\big(|X|^\alpha\big) = \int_{-\infty}^{\infty} \frac{|x|^\alpha}{\pi(1+x^2)}\,dx = \frac{2}{\pi}\int_0^{\infty} \frac{x^\alpha}{(1+x^2)}\,dx$$

since the integrand is an even function. Now for $0 < \alpha < 1$,

$$\int_0^{\infty} \frac{x^\alpha}{1+x^2}\,dx = \int_0^1 \frac{x^\alpha}{1+x^2}\,dx + \int_1^{\infty} \frac{x^\alpha}{1+x^2}\,dx.$$

Clearly, the first integral in the right side is convergent. To show that the second one is also convergent, note that.

$$\frac{x^\alpha}{1+x^2} \le \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}}.$$

Therefore,

$$\int_1^{\infty} \frac{x^\alpha}{1+x^2}\,dx \le \int_1^{\infty} \frac{1}{x^{2-\alpha}}\,dx = \left[\frac{1}{(\alpha-1)x^{1-\alpha}}\right]_1^{\infty} = \frac{1}{1-\alpha} < \infty.$$

For $\alpha \ge 1$,

$$\int_0^{\infty} \frac{x^\alpha}{1+x^2} \ge \int_1^{\infty} \frac{x^\alpha}{1+x^2}\,dx \ge \int_1^{\infty} \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\ln(1+x^2)\right]_1^{\infty} = \infty.$$

So $\displaystyle\int_0^{\infty} \frac{x^\alpha}{1+x^2}\,dx$ diverges.

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3. Let 2:00 P.M. be the origin, then a and b satisfy the following system of two equations in two unknown.

$$\begin{cases} \dfrac{a+b}{2} = 0 \\ \dfrac{(b-a)^2}{12} = 12. \end{cases}$$

Solving this system, we obtain $a = -6$ and $b = 6$. So the bus arrives at a random time between 1:54 P.M. and 2:06 P.M.

5. The probability density function of R, the radius of the sphere is

$$f(r) = \begin{cases} \dfrac{1}{4-2} = \dfrac{1}{2} & 2 < r < 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$E(V) = \int_2^4 \left(\frac{4}{3}\pi r^3\right)\frac{1}{2}\,dr = 40\pi.$$

$$P\left(\frac{4}{3}\pi R^3 < 36\pi\right) = P(R^3 < 27) = P(R < 3) = \frac{1}{2}.$$

10. Let F be the probability distribution function and f be the probability density function of X. By definition,

$$F(x) = P(X \le x) = P(\tan\theta \le x) = P(\theta \le \arctan x)$$

$$= \frac{\arctan x - \left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} = \frac{1}{\pi}\arctan x + \frac{1}{2}, \qquad -\infty < x < \infty.$$

Thus

$$f(x) = F'(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty.$$

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1. Since $np = (0.90)(50) = 45$ and $\sqrt{np(1-p)} = 2.12$,

$$P(X \ge 44.5) = P\left(Z \ge \frac{44.5 - 45}{2.12}\right) = P(Z \ge -0.24)$$

$$= 1 - \Phi(-0.24) = \Phi(0.24) = 0.5948.$$

5. $E(X\cos X)$, $E(\sin X)$, and $E\left(\dfrac{X}{1+X^2}\right)$ are, respectively, $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}(x\cos x)e^{-x^2/2}\,dx$, $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}(\sin x)e^{-x^2/2}\,dx$, and $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\dfrac{x}{1+x^2}e^{-x^2/2}\,dx$. Since these are integrals of *odd functions* from $-\infty$ to ∞, all three of them are 0.

9. $P(74.5 < X < 75.8) = P(-0.5 < Z < 0.8) = \Phi(0.8) - [1 - \Phi(0.5)] = 0.4796.$

15. Let X be the lifetime of a randomly selected light bulb.

$$P(X \geq 900) = P\Big(Z \geq \frac{900 - 1000}{100}\Big) = 1 - \Phi(-1) = \Phi(1) = 0.8413.$$

Hence the company's claim is false.

23. We have

$$\begin{aligned} E(e^{\alpha Z}) &= \int_{-\infty}^{\infty} e^{\alpha x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\ &= e^{\alpha^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2 + \alpha x - \frac{1}{2}x^2}\, dx \\ &= e^{\alpha^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}\, dx = e^{\alpha^2/2}, \end{aligned}$$

where $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}\, dx = 1$, since $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}$ is the probability density function of a normal random variable with mean α and variance 1.

33. Let X be the amount of soft drink in a random bottle. We are given that $P(X < 15.5) = 0.07$ and $P(X > 16.3) = 0.10$. These imply that $\Phi\Big(\frac{15.5 - \mu}{\sigma}\Big) = 0.07$ and and $\Phi\Big(\frac{16.3 - \mu}{\sigma}\Big) = 0.90$. Using Tables 1 and 2 of the appendix, we obtain

$$\begin{cases} \dfrac{15.5 - \mu}{\sigma} = -1.48 \\ \dfrac{16.3 - \mu}{\sigma} = 1.28. \end{cases}$$

Solving these two equations in two unknowns, we obtain $\mu = 15.93$ and $\sigma = 0.29$.

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1. Let X be the time until the next customer arrives; X is exponential with parameter $\lambda = 3$. Hence $P(X > x) = e^{-\lambda x}$, and $P(X > 3) = e^{-9} = 0.0001234$.

5. (a) Suppose that the next customer arrives in X minutes. By the memoryless property, the desired probability is

$$P\left(X < \frac{1}{30}\right) = 1 - e^{-5(1/30)} = 0.1535.$$

(b) Let Y be the time between the arrival times of the 10th and 11th customers; Y is exponential with $\lambda = 5$. So the answer is

$$P\left(Y \leq \frac{1}{30}\right) = 1 - e^{-5(1/30)} = 0.1535.$$

9. The answer is

$$E\left[350 - 40N(12)\right] = 350 - 40\left(\frac{1}{18} \cdot 12\right) = 323.33.$$