MATH ~~303~~ 30530, Fall 2009 – Homework 7 Solutions

Page 211:

1. $\lambda = (0.05)(60) = 3$; the answer is $1 - \dfrac{e^{-3}3^0}{0!} = 1 - e^{-3} = 0.9502$.

5. We call a room "success" if it is vacant next Saturday; we call it "failure" if it is occupied. Assuming that next Saturday is a random day, X, the number of vacant rooms on that day is approximately Poisson with rate $\lambda = 35$. Thus the desired probability is

$$1 - \sum_{i=0}^{29} \frac{e^{-35}(35)^i}{i!} = 0.823.$$

7. $P(X = 1) = P(X = 3)$ implies that $e^{-\lambda}\lambda = \dfrac{e^{-\lambda}\lambda^3}{3!}$ from which we get $\lambda = \sqrt{6}$. The answer is $\dfrac{e^{-\sqrt{6}}\left(\sqrt{6}\right)^5}{5!} = 0.063$.

9. Let X be the number of times the randomly selected kid has hit the target. We are given that $P(X = 0) = 0.04$; this implies that $\dfrac{e^{-\lambda}\lambda^0}{0!} = 0.04$ or $e^{-\lambda} = 0.04$. So $\lambda = -\ln 0.04 = 3.22$. Now

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) = 1 - 0.04 - \frac{e^{-\lambda}\lambda}{1!} \\ &= 1 - 0.04 - (0.04)(3.22) = 0.83. \end{aligned}$$

Therefore, 83% of the kids have hit the target at least twice.

11. Let $N(t)$ be the number of shooting stars observed up to time t. Let one minute be the unit of time. Then $\{N(t) : t \geq 0\}$ is a Poisson process with $\lambda = 1/12$. We have that

$$P\big(N(30) = 3\big) = \frac{e^{-30/12}(30/12)^3}{3!} = 0.21.$$

15. Choose one day as the unit of time. Then $\lambda = 3$ and the probability of no accidents in one day is

$$P\big(N(1) = 0\big) = e^{-3} = 0.0498.$$

The number of days without any accidents in January is approximately another Poisson random variable with approximate rate $31(0.05) = 1.55$. Hence the desired probability is

$$\frac{e^{-1.55}(1.55)^3}{3!} \approx 0.13.$$

17. The expected number of fractures per meter is $\lambda = 1/60$. Let $N(t)$ be the number of fractures in t meters of wire. Then

$$P\big(N(t) = n\big) = \frac{e^{-t/60}(t/60)^n}{n!}, \quad n = 0, 1, 2, \ldots .$$

In a ten minute period, the machine turns out 70 meters of wire. The desired probability, $P\big(N(70) > 1\big)$ is calculated as follows:

$$\begin{aligned} P\big(N(70) > 1\big) &= 1 - P\big(N(70) = 0\big) - P\big(N(70) = 1\big) \\ &= 1 - e^{-70/60} - \frac{70}{60}e^{-70/60} \approx 0.325. \end{aligned}$$

19. Let X be the number of earthquakes of magnitude 5.5 or higher on the Richter scale during the next 60 years. Clearly, X is a Poisson random variable with parameter $\lambda = 6(1.5) = 9$. Let A be the event that the earthquakes will not damage the bridge during the next 60 years. Since the events $\{X = i\}$, $i = 0, 1, 2, \ldots$, are mutually exclusive and $\bigcup_{i=1}^{\infty}\{X = i\}$ is the sample space, by the Law of Total Probability (Theorem 3.4),

$$\begin{aligned} P(A) &= \sum_{i=0}^{\infty} P(A \mid X = i)P(X = i) = \sum_{i=0}^{\infty}(1 - 0.015)^i \frac{e^{-9}\,9^i}{i!} \\ &= \sum_{i=0}^{\infty}(0.985)^i \frac{e^{-9}\,9^i}{i!} = e^{-9}\sum_{i=0}^{\infty}\frac{\big[(0.985)(9)\big]^i}{i!} = e^{-9}e^{(0.985)(9)} = 0.873716. \end{aligned}$$