MATH 30530 FALL 2009 Homework 12 Solns

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1. $M_X(t) = E(e^{tX}) = \sum_{x=1}^{5} e^{tx} p(x) = \frac{1}{5}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t}).$

3. Note that

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \cdot 2\left(\frac{1}{3}\right)^x = 2\sum_{x=1}^{\infty} e^{tx} \cdot e^{-x\ln 3} = 2\sum_{x=1}^{\infty} e^{x(t-\ln 3)}.$$

Restricting the domain of $M_X(t)$ to the set $\{t : t < \ln 3\}$ and using the geometric series theorem, we get

$$M_X(t) = 2\left(\frac{e^{t-\ln 3}}{1 - e^{t-\ln 3}}\right) = \frac{2e^t}{3 - e^t}.$$

(Note that $e^{-\ln 3} = 1/3$.) Differentiating $M_X(t)$, we obtain

$$M_X'(t) = \frac{6e^t}{(3 - e^t)^2},$$

which gives $E(X) = M_X'(0) = 3/2$.

8. The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $t \neq 0$,

$$M_X(t) = E(e^{tX}) = \int_a^b \frac{1}{b-a} e^{tx}\,dx = \frac{1}{b-a}\left(\frac{e^{tb} - e^{ta}}{t}\right),$$

whereas for $t = 0$, $M_X(0) = 1$. Thus

$$M_X(t) = \begin{cases} \dfrac{1}{b-a}\left(\dfrac{e^{tb} - e^{ta}}{t}\right) & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

9. The probability mass function of a geometric random variable X, $p(x)$ with parameter p is given by

$$p(x) = pq^{x-1}, \quad q = 1 - p, \quad x = 1, 2, 3, \ldots.$$

Thus

$$M_X(t) = \sum_{x=1}^{\infty} pq^{x-1}e^{tx} = \frac{p}{q}\sum_{x=1}^{\infty}\left(qe^t\right)^x.$$

Now by the geometric series theorem, $\sum_{x=1}^{\infty}\left(qe^t\right)^x$ converges to $\left(qe^t\right)/\left(1 - qe^t\right)$ if $qe^t < 1$ or, equivalently, if $t < -\ln q$. Restricting the domain of $M_X(t)$ to the set $\{t : t < -\ln q\}$, we obtain

$$M_X(t) = \frac{p}{q}\sum_{x=1}^{\infty}\left(qe^t\right)^x = \frac{p}{q}\cdot\frac{qe^t}{1 - qe^t} = \frac{pe^t}{1 - qe^t}.$$

Now

$$M_X'(t) = \frac{pe^t}{(1 - qe^t)^2} \quad \text{and} \quad M_X''(t) = \frac{pe^t + pqe^{2t}}{(1 - qe^t)^3}.$$

Therefore,

$$E(X) = M_X'(0) = \frac{p}{(1 - q)^2} = \frac{1}{p}.$$

and

$$E(X^2) = M_X''(0) = \frac{p(1 + q)}{(1 - q)^3} = \frac{1 + q}{p^2}.$$

Thus

$$\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2 = \frac{1 + q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

13. Note that

$$M_X'(t) = \frac{24}{(2 - t)^4}, \quad M_X''(t) = \frac{96}{(2 - t)^5}.$$

Therefore,

$$E(X) = M_X'(0) = \frac{24}{16} = \frac{3}{2}, \quad E(X^2) = M_X''(0) = \frac{96}{32} = 3,$$

and hence $\mathrm{Var}(X) = 3 - (9/4) = 3/4$.

15. For a random variable X, we must have $M_X(0) = 1$. Since $t/(1 - t)$ is 0 at 0, it cannot be a moment-generating function.

19. We know that for $t \neq 0$,

$$M_X(t) = \frac{e^t - 1}{t(1 - 0)} = \frac{e^t - 1}{t}.$$

Therefore, for $t \neq 0$,

$$M_{aX+b}(t) = E\left[e^{t(aX+b)}\right] = e^{bt}E\left[e^{atX}\right] = e^{bt}M_X(at)$$

$$= e^{bt}\cdot\frac{e^{at} - 1}{at} = \frac{e^{(a+b)t} - e^{bt}}{\left[(a + b) - b\right]t},$$

which is the moment-generating function of a uniform random variable over $(b, a + b)$.

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1. $M_{\alpha X}(t) = E\left(e^{t\alpha X}\right) = M_X(t\alpha) = \exp\left[\alpha\mu t + (1/2)\alpha^2\sigma^2t^2\right].$

3. Since

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n,$$

$X_1 + X_2 + \cdots + X_n$ is gamma with parameters n and λ.

10. By Theorem 11.7, $X + Y \sim N(5, 9)$, $X - Y \sim N(-3, 9)$, and $3X + 4Y \sim N(19, 130)$. Thus

$$P(X + Y > 0) = P\left(\frac{X + Y - 5}{3} > \frac{0 - 5}{3}\right) = 1 - \Phi(-1.67) = \Phi(1.67) = 0.9525,$$

$$P(X - Y < 2) = P\left(\frac{X - Y + 3}{3} < \frac{2 + 3}{3}\right) = \Phi(1.67) = 0.9525,$$

and

$$P(3X + 4Y > 20) = P\left(\frac{3X + 4Y - 19}{\sqrt{130}} > \frac{20 - 19}{\sqrt{130}}\right) = 1 - \Phi(0.9) = 0.4641.$$

11. Theorem 11.7 implies that $\bar{X} \sim N(110, 1.6)$, where $\bar{X}$ is the average of the IQ's of the randomly selected students. Therefore,

$$P(\bar{X} \geq 112) = P\left(\frac{\bar{X} - 110}{\sqrt{1.6}} \geq \frac{112 - 110}{\sqrt{1.6}}\right) = 1 - \Phi(1.58) = 0.0571.$$

15. Let $\bar{X}$ be the average of the weights of the 12 randomly selected athletes. Let $X_1, X_2, \ldots, X_{12}$ be the weights of these athletes. Since

$$\bar{X} \sim N\left(225, \frac{25^2}{12}\right) = N\left(225, \frac{625}{12}\right),$$

we have that

$$P(X_1 + X_2 + \cdots + X_{12} \leq 2700) = P\left(\bar{X} \leq \frac{2700}{12}\right) = P(\bar{X} \leq 225)$$

$$= P\left(\frac{\bar{X} - 225}{\sqrt{625/12}} \leq \frac{225 - 225}{\sqrt{625/12}}\right) = \Phi(0) = \frac{1}{2}.$$

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3. We have that

$$\mu = \int_1^3 \frac{1}{9}x\left(x+\frac{5}{2}\right)dx = \frac{56}{27} = 2.07,$$

$$E(X^2) = \int_1^3 \frac{1}{9}x^2\left(x+\frac{5}{2}\right)dx = \frac{125}{27},$$

$$\sigma_X = \sqrt{(125/27) - (56/27)^2} = 0.57.$$

The desired probability is

$$P(2 < \bar{X} < 2.15) = P\left(2 < \frac{X_1 + X_2 + \cdots + X_{24}}{24} < 2.15\right)$$

$$= P(48 < X_1 + X_2 + \cdots + X_{24} < 51.6)$$

$$= P\left(\frac{48 - 24(2.07)}{0.57\sqrt{24}} < \frac{X_1 + X_2 + \cdots + X_{24} - 24(2.07)}{0.57\sqrt{24}} < \frac{51.6 - 24(2.07)}{0.57\sqrt{24}}\right)$$

$$\approx \Phi(0.69) - \Phi(-0.60) = 0.7549 - 0.2743 = 0.4806.$$

7. Note that *actual value* is a nebulous concept. In this exercise, like everywhere else, we are using it to mean the average of a *very large* number of measurements. Let X_i be the error in the ith measurement; $\mu = E(X_i) = 0$, $\sigma = \sigma_{X_i} = 1/\sqrt{3}$. Hence

$$P\left(-0.25 < \frac{X_1 + X_2 + \cdots + X_{50}}{50} < 0.25\right)$$

$$= P(-12.5 < X_1 + X_2 + \cdots + X_{50} < 12.5)$$

$$= P\left(\frac{-12.5}{(1/\sqrt{3})\sqrt{50}} < \frac{X_1 + X_2 + \cdots + X_{50}}{(1/\sqrt{3})\sqrt{50}} < \frac{12.5}{(1/\sqrt{3})\sqrt{50}}\right)$$

$$\approx \Phi(3.06) - \Phi(-3.06) = 2\Phi(3.06) - 1 = 0.9778.$$

13. Let $Y_n = \sum_{i=1}^n X_i$; Y_n is Poisson with rate n. On the one hand,

$$P(Y_n \le n) = \sum_{k=0}^{n} \frac{e^{-n}n^k}{k!} = \frac{1}{e^n}\sum_{k=0}^{n}\frac{n^k}{k!},$$

and on the other hand,

$$\lim_{n\to\infty} P(Y_n \le n) = \lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i \le n\right)$$

$$= \lim_{n\to\infty} P\left(\frac{\sum_{i=1}^n X_i - n}{\sqrt{n}} \le \frac{n-n}{\sqrt{n}}\right) = \Phi(0) = \frac{1}{2}.$$

So

$$\lim_{n\to\infty} \frac{1}{e^n}\sum_{k=0}^{\infty}\frac{n^k}{k!} = \frac{1}{2}.$$