MATH 30530, Homework 11 Solutions

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1. Note that $p_X(x) = (1/25)(3x^2+5)$, $p_Y(y) = (1/25)(2y^2+5)$. Now $p_X(1) = 8/25$, $p_Y(0) = 5/25$, and $p(1,0) = 1/25$. Since $p(1,0) \neq p_X(1)p_Y(0)$, X and Y are dependent.

3. By the independence of X and Y,

$$P(X=1,\ Y=3) = P(X=1)P(Y=3) = \frac{1}{2}\Big(\frac{2}{3}\Big)\cdot\frac{1}{2}\Big(\frac{2}{3}\Big)^3 = \frac{4}{81}.$$

$$\begin{aligned}P(X+Y=3) &= P(X=1,\ Y=2) + P(X=2,\ Y=1)\\ &= \frac{1}{2}\Big(\frac{2}{3}\Big)\cdot\frac{1}{2}\Big(\frac{2}{3}\Big)^2 + \frac{1}{2}\Big(\frac{2}{3}\Big)^2\cdot\frac{1}{2}\Big(\frac{2}{3}\Big) = \frac{4}{27}.\end{aligned}$$

6. We have that

$$P\big(\max(X,Y)\le t\big) = P(X\le t,\ Y\le t) = P(X\le t)P(Y\le t) = F(t)G(t).$$

$$\begin{aligned}P\big(\min(X,Y)\le t\big) &= 1 - P\big(\min(X,Y) > t\big)\\ &= 1 - P(X>t,\ Y>t) = 1 - P(X>t)P(Y>t)\\ &= 1 - \big[1-F(t)\big]\big[1-G(t)\big] = F(t)+G(t)-F(t)G(t).\end{aligned}$$

11. We have that

$$f_X(x) = \int_0^\infty x^2 e^{-x(y+1)}\,dy = xe^{-x}, \quad x\ge 0;$$

$$f_Y(y) = \int_0^\infty x^2 e^{-x(y+1)}\,dx = \frac{2}{(y+1)^3}, \quad y\ge 0,$$

where the second integral is calculated by applying integration by parts twice. Now since $f(x,y)\neq f_X(x)f_Y(y)$, X and Y are not independent.

13. Since

$$f(x,y) = e^{-x}\cdot 2e^{-2y} = f_X(x)f_Y(y),$$

X and Y are independent exponential random variables with parameters 1 and 2, respectively. Therefore,

$$E(X^2Y) = E(X^2)E(Y) = 2\cdot\frac{1}{2} = 1.$$

15. Let F and f be the probability distribution and probability density functions of $\max(X, Y)$, respectively. Clearly,

$$F(t) = P\big(\max(X, Y) \leq t\big) = P(X \leq t,\ Y \leq t) = (1 - e^{-t})^2, \quad t \geq 0.$$

Thus

$$f(t) = F'(t) = 2e^{-t}(1 - e^{-t}) = 2e^{-t} - 2e^{-2t}.$$

Hence

$$E\big[\max(X, Y)\big] = 2\int_0^{\infty} te^{-t}\,dt - \int_0^{\infty} 2te^{-2t}\,dt = 2 - \frac{1}{2} = \frac{3}{2}.$$

Note that $\int_0^{\infty} te^{-t}\,dt$ is the expected value of an exponential random variable with parameter 1, thus it is 1. Also, $\int_0^{\infty} 2te^{-2t}\,dt$ is the expected value of an exponential random variable with parameter 2, thus it is 1/2.

22. The joint probability density function of B and C is

$$f(b, c) = \begin{cases} \dfrac{9b^2c^2}{676} & 1 < b < 3,\ 1 < c < 3 \\ 0 & \text{otherwise.} \end{cases}$$

For $X^2 + BX + C$ to have two real roots we must have $B^2 - 4C > 0$, or, equivalently, $B^2 > 4C$. Let

$$E = \big\{(b, c) \colon 1 < b < 3,\ 1 < c < 3,\ b^2 > 4c\big\};$$

the desired probability is

$$\iint_E \frac{9b^2c^2}{676}\,db\,dc = \int_2^3 \Big(\int_1^{b^2/4} \frac{9b^2c^2}{676}\,dc\Big)\,db \approx 0.12.$$

(Draw a figure to verify the region of integration.)

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4. Since $f(x, y) = e^{-x} \cdot 2e^{-2y}$, X and Y are independent exponential random variables with parameters 1 and 2, respectively. Thus $E(X) = 1$, $E(Y) = 1/2$,

$$E(X^2) = \text{Var}(X) + \big[E(X)\big]^2 = 1 + 1 = 2,$$

and

$$E(Y^2) = \text{Var}(Y) + \big[E(Y)\big]^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, $E(X^2 + Y^2) = 2 + \dfrac{1}{2} = \dfrac{5}{2}.$

5. Let X_1, X_2, X_3, X_4, and X_5 be geometric random variables with parameters 1, 4/5, 3/5, 2/5, and 1/5, respectively. The desired quantity is

$$E(X_1+X_2+X_3+X_4+X_5) = E(X_1)+E(X_2)+E(X_3)+E(X_4)+E(X_5)$$
$$= 1+\frac{5}{4}+\frac{5}{3}+\frac{5}{2}+5 = 11.42.$$

11. Let E_1 be the event that the first three outcomes are heads and the fourth outcome is tails. For $2 \le i \le n-3$, let E_i be as defined in the hint. Let E_{n-2} be the event that the outcome $(n-3)$ is tails and the last three outcomes are heads. The expected number of exactly three consecutive heads is

$$E\Big(X_1+\sum_{i=2}^{n-3} X_i + X_{n-2}\Big) = E(X_1)+\sum_{i=2}^{n-3} E(X_i)+E(X_{n-2})$$
$$= P(E_1)+\sum_{i=2}^{n-3} P(E_i)+P(E_{n-2})$$
$$= \Big(\frac{1}{2}\Big)^4+\sum_{i=2}^{n-3}\Big(\frac{1}{2}\Big)^5+\Big(\frac{1}{2}\Big)^4$$
$$= \Big(\frac{1}{2}\Big)^3+(n-4)\Big(\frac{1}{2}\Big)^5 = \frac{n}{32}.$$

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2. $E(X) = \displaystyle\sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}x^2(x+y) = \frac{17}{7};$

$$E(Y) = \textstyle\sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}xy(x+y) = \frac{124}{35};$$

$$E(XY) = \sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}x^2y(x+y) = \frac{43}{5}.$$

Therefore,

$$\mathrm{Cov}(X,Y) = E(XY)-E(X)E(Y) = \frac{43}{5}-\frac{17}{7}\cdot\frac{124}{35} = -\frac{1}{245}.$$

3. Intuitively, $E(X)$ is the average of $1, 2, \ldots, 6$ which is 7/2; $E(Y)$ is $(7/2)(1/2) = 7/4$. To show these, note that

$$E(X) = \sum_{x=1}^{6} xp_X(x) = \sum_{x=1}^{6} x(1/6) = 7/2.$$

By the table constructed for $p(x, y)$ in Example 8.2,

$$E(Y) = 0 \cdot \frac{63}{384} + 1 \cdot \frac{120}{384} + 2 \cdot \frac{99}{384} + 3 \cdot \frac{64}{384} + 4 \cdot \frac{29}{384} + 5 \cdot \frac{8}{384} + 6 \cdot \frac{1}{384} = \frac{7}{4}.$$

By the same table,

$$E(XY) = \sum_{x=1}^{6} \sum_{y=0}^{6} xyp(x, y) = 91/12.$$

Therefore,

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{91}{12} - \frac{7}{2} \cdot \frac{7}{4} = \frac{35}{24} > 0.$$

This shows that X and Y are positively correlated. The higher the outcome from rolling the die, the higher the number of tails obtained—a fact consistent with our intuition.

7. Since $\mathrm{Cov}(X, Y) = 0$, we have

$$\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z) = \mathrm{Cov}(X, Z).$$

14. We have that

$$\begin{aligned}\mathrm{Var}(XY) &= E(X^2Y^2) - \big[E(X)E(Y)\big]^2 = E(X^2)E(Y^2) - \mu_1^2\mu_2^2 \\ &= (\mu_1^2 + \sigma_1^2)(\mu_2^2 + \sigma_2^2) - \mu_1^2\mu_2^2 = \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2.\end{aligned}$$

15. (a) Let U_1 and U_2 be the measurements obtained using the voltmeter for V_1 and V_2, respectively. Then $V_1 = U_1 + X_1$ and $V_2 = U_2 + X_2$, where X_1 and X_2, the measurement errors, are independent random variables with mean 0 and variance σ^2. So the error variance in the estimation of V_1 and V_2 using the first method is σ^2.

(b) Let U_3 and U_4 be the measurements obtained,using the voltmeter, for V and W, respectively. Then $V = U_3 + X_3$ and $W = U_4 + X_4$, where X_3 and X_4, the measurement errors, are independent random variables with mean 0 and variance σ^2. Since $(U_3 + U_4)/2$ is used to estimate V_1, and $(U_3 - U_4)/2$ is used to estimate V_2,

$$V_1 = \frac{V + W}{2} = \frac{U_3 + U_4}{2} + \frac{X_3 + X_4}{2},$$

and

$$V_2 = \frac{V - W}{2} = \frac{U_3 - U_4}{2} + \frac{X_3 - X_4}{2},$$

we have that, for part (b), $(X_3 + X_4)/2$ and $(X_3 - X_4)/2$ are the measurement errors in measuring V_1 and V_2, respectively. The independence of X_3 and X_4 yields

$$\mathrm{Var}\Big(\frac{X_3 + X_4}{2}\Big) = \frac{1}{4}\big[\mathrm{Var}(X_3) + \mathrm{Var}(X_4)\big] = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{\sigma^2}{2},$$

and

$$\mathrm{Var}\Big(\frac{X_3 - X_4}{2}\Big) = \frac{1}{4}\big[\mathrm{Var}(X_3) + \mathrm{Var}(X_4)\big] = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{\sigma^2}{2}.$$

Therefore, the error variances in the estimation of V_1 and V_2, using the second method, is $\sigma^2/2$, showing that the second method is preferable.

19. For $1 \le i \le n$, let X_i be the ith random number selected; we have

$$\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{n} \frac{(1-0)^2}{12} = \frac{n}{12}.$$

27. To find $\text{Var}(X)$, we use the following identity:

$$\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X_i, X_j). \tag{38}$$

Now for $1 \le i \le n$,

$$E(X_i) = P(A_i) = \frac{D}{N}, \qquad E(X_i^2) = P(A_i) = \frac{D}{N}.$$

Thus

$$\text{Var}(X_i) = E(X_i^2) - \big[E(X_i)\big]^2 = \frac{D}{N} - \Big(\frac{D}{N}\Big)^2 = \frac{D(N-D)}{N^2}.$$

Also for $i < j$,

$$X_i X_j = \begin{cases} 1 & \text{if } A_i A_j \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(X_i X_j) = P(A_i A_j) = P(A_j \mid A_i)P(A_i) = \frac{D-1}{N-1} \cdot \frac{D}{N} = \frac{(D-1)D}{(N-1)N},$$

and

$$\begin{aligned}\text{Cov}(X_i, X_j) &= E(X_i X_j) - E(X_i)E(X_j) \\ &= \frac{(D-1)D}{(N-1)N} - \frac{D}{N} \cdot \frac{D}{N} = \frac{-D(N-D)}{(N-1)N^2}.\end{aligned}$$

Substituting the values of $\text{Var}(X_i)$'s and $\text{Cov}(X_i, X_j)$ back into (38), we get

$$\begin{aligned}\text{Var}(X) &= n\Big[\frac{D(N-D)}{N^2}\Big] + 2\binom{n}{2}\Big[\frac{-D(N-D)}{(N-1)N^2}\Big] \\ &= \frac{nD(N-D)}{N^2}\Big(1 - \frac{n-1}{N-1}\Big).\end{aligned}$$

This follows since in (38), $\sum$ and $\sum\sum_{i<j}$ have n and $\binom{n}{2} = \frac{n(n-1)}{2}$ equal terms, respectively.

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2. By Exercise 23 of Section 8.2, X and Y are independent random variables. [This can also be shown directly by verifying the relation $f(x, y) = f_X(x)f_Y(y)$.] Hence $\text{Cov}(X, Y) = 0$, and therefore $\rho(X, Y) = 0$.