Introduction to Probability, Fall 2009

Math 30530

Review questions for exam 1 — solutions

1. Let A, B and C be events. Some of the following statements are always true, and some are not. For those that are always true, give a proof. For those that are sometimes false, use the experiment of picking a random integer between 1 and 4 to give examples to show that they are sometimes false and sometimes true.

   (a) $P(A \cup B) = P(A) + P(B)$.
   Solution: Not always true; eg $A = \{1, 2, 3\}, B = \{2, 3, 4\}$.

   (b) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.
   Solution: Always true: $A \cup B \cup C = A \cup B - A \cup C - (A \cup B)$, and the latter union is disjoint, so $P(A \cup B \cup C) = P(A) + P(B - A) + P(C - (A \cup B))$. Since $B - A \subseteq B$ and $C - (A \cup B) \subseteq C$, this is at most $P(A) + P(B) + P(C)$.

   (c) $P(AC) = P(A)P(C)$.
   Solution: Not always true; eg $A = \{1, 2, 3\}, B = \{2, 3, 4\}$.

2. Two students miss a midterm because they were at a party at Purdue that ran way late. They tell the professor that the reason they missed the exam was because they were driving together to school and had a flat tire. The professor agrees to give them a make-up exam. He puts them in separate rooms and gives them each a sheet of paper on which is written the question "Which tire?". What's the probability that they both give the same answer? (Assume that a car has 4 tires).

   Solution: $1/4$ (not $1/16$!). There are 16 sample points (each of the students has four choices), in 4 of which they agree.

3. A number is selected at random from the set $\{1, 2, 3, \ldots, 150\}$. Use the principle of inclusion-exclusion to find the probability that the chosen number has no prime factors in common with 150. (The prime factors of 150 are 2, 3 and 5.)

   Solution: 75 numbers divisible by 2, 50 divisible by 3, 30 divisible by 5, 25 divisible by 2 and 3, 15 divisible by 2 and 5, 10 divisible by 3 and 5, 5 divisible by 2, 3 and 5, so by inclusion-exclusion the number divisible by at least one of the three is

   $$75 + 50 + 30 - 25 - 15 - 10 + 5 = 110,$$

   and the number divisible by none of the three is 40; the probability is $40/150 = 4/15$.

4. If four different fair dice are tossed, what is the probability that they will show four different numbers?

   Solution: Number of outcomes: 6^4. Number of outcomes with 4 different numbers: $6.5.4.3$. Probability is $6.5.4.3/6^4 = 5/18$.

5. In how many arrangements of the letters "WEWILLWHALEONUSC" are the three W's adjacent?

   Solution: 16 letters, so 14 places where the string of 3 W's can begin. For each each such place, 3! ways to order the W's and 13! ways to arrange the remaining letters. So $14.3!.13! = 523069747200$.

6. When entering data into a program, I mistakenly enter as negative two numbers from among six numbers that should all have been entered as positive, and I mistakenly enter as positive two numbers from among four numbers that should all have been entered as negative. If at some point the program chooses three distinct numbers from among these 10 and multiplies them, what is the probability that at this stage no mistake occurs?

   Solution: $\binom{10}{3}$ possible choices. Of the ten numbers, 4 are wrong by a sign, so no mistake will occur if the computer chooses an even number — 0 or 2 — of wrong entries. Number of choices with 0 from among the four is $\binom{7}{3}$, and number of choices with 2 from among the four is $6\binom{4}{2}$. So the probability is

   $$\frac{\binom{7}{3} + 6\binom{4}{2}}{\binom{10}{3}} = \frac{71}{120} = .5916...$$

7. 98% of the time in which a person is telling the truth, a lie detector correctly detects this. 95% of the time in which a person is lying, the lie detector correctly detects this. An operator knows from experience that when suspects are being interrogated, they lie two thirds of the time. Based on this information, what would you say is the probability that a statement detected as a lie is in fact a lie? (Let D be the event that a statement is detected as a lie, T the event that a statement is not a lie and L the event that a statement is a lie).

   Solution:

   $$P(L|D) = \frac{P(D|L)P(L)}{P(D|L)P(L) + P(D|T)P(T)} = \frac{.95 \times .66...}{.95 \times .66... + .02 \times .33...} = .9895...$$

8. My family has three cars. On cold winter days, the Porsche will start with probability .9, the Ferrari with probability .8 and the Edsel with probability .7, each independently of the others.

   (a) On a given cold morning, what is the probability that at least two cars will start, allowing both my wife and I to drive to work?

   Solution: $.9 \times .8 \times .3 + .9 \times .2 \times .7 + .1 \times .8 \times .7 + .9 \times .8 \times .7 = .902$. (First term is probability that Porsche and Ferrari start and Edsel does not, etc.)

   (b) My wife goes in to the garage and comes back to tell me that she has tried one randomly chosen car, and it starts. I know that she is twice as likely to have tried the Porsche as the Edsel, and twice as likely to have tried the Edsel as the Porsche. What is now the probability that at least two cars will start?

   Solution: Let P, F and E be the events that my wife choose to test the Porsche, etc. and finds it to start, and T the event that at least two cars start, given that my wife has given me the information that the car she has tested starts. We have

   $$P(P) = 2P(F), \quad P(F) = 2P(E), \quad P(P) + P(F) + P(E) = 1$$

   from which we determine $P(P) = 4/7$, $P(F) = 2/7$, $P(E) = 1/7$. Also,

   $$P(T|E) = .9 \times .2 + .1 \times .8 + .9 \times .8 = .98$$

(at least one of the Porsche, Ferrari has to start) and similarly, $P(T|P) = .94, P(T|F) = .97$. By law of total probability, the probability that at least two cars start is

$$P(T|E)P(E) + P(T|P)P(P) + P(T|F)P(F) = .98 \times \frac{1}{7} + .94 \times \frac{4}{7} + .97 \times \frac{2}{7} = .954...$$

9. A woman gives birth to a child in the hospital's maternity ward. After a while, the child is brought into the hospital's nursery. Before the child is brought in, there are 5 boys and 10 girls in the nursery. Sometime after the child is brought in, a doctor walks into the nursery, picks a child at random, and notices that it's a boy. What's the probability that the woman gave birth to a boy? (You should assume that when a child is born, the probability that it is a boy is .5 and the probability that it is a girl is .5.)

   Solution: B is boy, G girl, OB is event doctor observes boy. We want

$$P(B|OB) = \frac{P(OB|B)P(B)}{P(OB|B)P(B) + P(OB|G)P(G)} = \frac{(6/16) \times .5}{(6/16) \times .5 + (5/16) \times .5} = .5454...$$

10. From a regular deck of 52 cards, ten are drawn at random without replacement. If exactly four of them are hearts, what is the probability that one of the others is a spade?

    Solution: Probability of *no* spade given 4 hearts is P(no spade AND 4 hearts)/P(4 hearts). This is

$$\frac{\frac{\binom{13}{4}\binom{26}{6}}{\binom{52}{10}}}{\frac{\binom{13}{4}\binom{39}{6}}{\binom{52}{10}}} = \frac{\binom{26}{6}}{\binom{39}{6}}$$

    so P(at least one spade given 4 hearts) is

$$1 - \frac{\binom{26}{6}}{\binom{39}{6}} = .9294...$$

    (Could also think of this via collapsed sample space: given *exactly* 4 hearts, new experiment is to draw six cards 39, and we want to know what's the probability that we avoid the 13 spades.)

11. A certain component in a (shoddy) computer typically fails 30% of the time, causing the computer to break. To counteract this appalling problem, a hacker decides to install n copies of the component in parallel, in such a way that the computer only breaks if all n components fail at the same time. Assuming that component failures are independent of each other,

    (a) find the probability that the computer does not break if $n = 3$ and
        Solution: $1 - .3^3 = .973$

    (b) find the smallest value of n that should be chosen to ensure that the probability that the computer does not break is at least 98%.
        Solution: $n = 3$ not good enough (from part a), but $n = 4$ works: $1 - .3^4 = .9919$

12. 2% of all people who are qualified to apply for a position as administrative assistant at a mathematics department are familiar with the typesetting language LaTeX. How many qualified applicants should a department interview if it wants to be 50% sure that at least one of the applicants is familiar with LaTeX?

    Solution: If department interviews n applicants, the probability that they all fail to know LaTeX is $.98^n$. We want n large enough so that this is below .5. $n = 35$ is smallest that will work.

13. I have prepared 12 exams, 4 of which are hard and 8 easy. I select 3 of them at random to give to the three sections of the courses I am teaching this semester. Let X be the number of sections that get a hard test.

   (a) Compute the mass function of X.

   Solution:

   $$\begin{aligned}
   P(X = 0) &= \binom{8}{3} / \binom{12}{3} = 14/55 \\
   P(X = 1) &= \binom{8}{2}\binom{4}{1} / \binom{12}{3} = 28/55 \\
   P(X = 2) &= \binom{8}{1}\binom{4}{2} / \binom{12}{3} = 12/55 \\
   P(X = 3) &= \binom{4}{3} / \binom{12}{3} = 1/55
   \end{aligned}$$

   (b) Compute the expectation of X.

   Solution: $1 \times (28/55) + 2 \times (12/55) + 3 \times (1/55) = 1$

   (c) Compute the variance of X.

   Solution: $(0-1)^2 \times (14/55) + (1-1)^2 \times (28/55) + (2-1)^2 \times (12/55) + (3-1)^2 \times (1/55) = 30/55$

14. I toss a fair coin repeatedly until I have seen either 4 Heads or 4 Tails. Let X be the random variable which measures the number of times I have to toss the coin (so, for example, $X(HTHTHTH) = 7$). This is intended as a simple model of the number of games in an evenly-matched World Series.

   (a) Compute the mass function of X.

   Solution:

   $$\begin{aligned}
   P(X = 4) &= \frac{2}{16} = .125 \quad (HHHH, TTTT) \\
   P(X = 5) &= \frac{8}{32} = .25 \quad (THHHH, HTHHH, HHTHH, HHHTH, 4 \text{ more with } T \text{ winning}) \\
   P(X = 6) &= \frac{20}{64} = .3125 \quad (10 \text{ ways for } T \text{ to win 2 of first 5 and lose}) \\
   P(X = 7) &= \frac{40}{128} = .3125 \quad (20 \text{ ways for } T \text{ to win 3 of first 6 and lose})
   \end{aligned}$$

   (b) Compute the expectation of X.

   Solution: $E(X) = 4 \times .125 + 5 \times .25 + 6 \times .3125 + 7 \times .3125 = 5.8125$

15. Experience has shown that the number of people who enter a certain office building per minute during the middle of a quiet day is a random variable X with mass function

$$p(x) = \frac{1}{3}c^x, x = 0, 1, 2, 3, \ldots.$$

   (a) What is c?

   Solution: Need $\sum_{x \geq 0} c^x / 3 = 1$, so $1/(1 - c) = 3$, $c = 2/3$.

(b) What is the probability that between 2 and 5 people (inclusive) enter the building in a given minute?

Solution: $(1/3)(2/3)^2 + (1/3)(2/3)^3 + (1/3)(2/3)^4 + (1/3)(2/3)^5 = .3566...$

16. A bag contains 30 balls numbered 1 through 30. Seven balls are selected at random, one at a time, with replacement. What is the probability that exactly four of the selected balls have prime numbers on them?

Solution: There are 10 primes 30 or below (2, 3, 5, 7, 11, 13, 17, 19, 23, 29), so this is binomial trial with $p = 1/3$, $n = 7$, and we seek the probability that answer is 4:

$$\binom{7}{4}(1/3)^4(2/3)^3 = .128...$$