# MATH 30440, SEC 01, SPRING 2010

# EXAM 1 PRACTICE SOLUTIONS

1) a) $\int_4^\infty \frac{c}{x^4}\,dx = \left[-\frac{c}{3}\frac{1}{x^3}\right]_4^\infty = \frac{c}{192}$

So $c = 192$

b) $E(X) = \int_4^\infty \frac{192}{x^3}\,dx = \left[-\frac{96}{x^2}\right]_4^\infty = 6$

$E(X^2) = \int_4^\infty \frac{192}{x^2}\,dx = \left[-\frac{192}{x}\right]_4^\infty = 48$

$Var(X) = 48 - (6)^2 = 12$

c) $P(X > t) = \int_t^\infty \frac{192}{x^4}\,dx = \left[-\frac{64}{x^3}\right]_t^\infty = \frac{64}{t^3}$

Want $= \frac{1}{2}$; so $t = \sqrt[3]{32}$

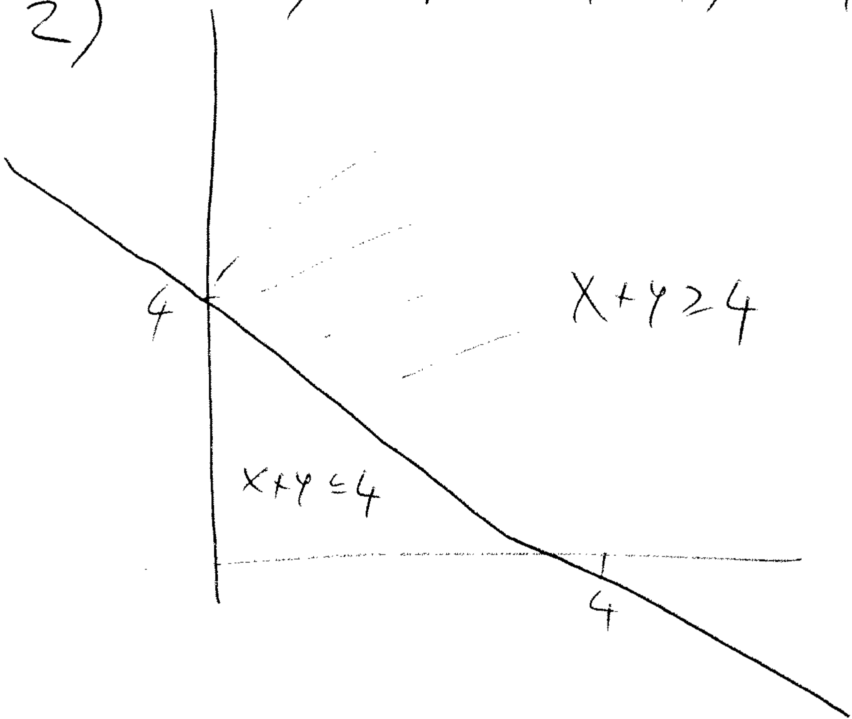
d) $E(X^2+X) = \int_4^\infty (x^2+x)\frac{192}{x^4}\,dx$

$= 192\int_4^\infty \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx$

$= 192\left[-\frac{1}{x} - \frac{1}{2x^2}\right]_4^\infty$

$= 192\left[\frac{1}{4} + \frac{1}{32}\right] = 54$

2) a) $P(X+Y \geq 4) = 1 - P(X+Y \leq 4)$




$$P(X+Y \leq 4) = \int_0^4 \int_0^{4-x} 6e^{-2x}e^{-3y}\,dy\,dx$$

$$= \int_0^4 6e^{-2x}\left[-\tfrac{1}{3}e^{-3y}\right]_0^{4-x} dx$$

$$= \int_0^4 6e^{-2x}\left[1 - \tfrac{1}{3}e^{-12+3x}\right] dx$$

$$= \int_0^4 6e^{-2x}\,dx - \int_0^4 2e^{-12+3x}\,dx$$

$$= \left[-3e^{-2x}\right]_0^4 - 2e^{-12}\left[\tfrac{1}{3}e^{3x}\right]_0^4 dx$$

$$= 1 - 3e^{-8} - \tfrac{2}{3} + \tfrac{2}{3}e^{-12}$$

So $P(X+Y \geq 4) = 3e^{-8} + \frac{2}{3} - \frac{2}{3}e^{-12}$.

b) $E(X+Y) = \int_0^\infty \int_0^\infty (x+y)\, 6e^{-2x} e^{-3y}\, dx\, dy$

3) $\dfrac{6 \times 5 \times 4 \times 3}{6^4}$ $\begin{array}{l} \leftarrow \text{ways to get 4 different \#s} \\ \leftarrow \text{total \# of outcomes} \end{array}$

4) $P(\text{Elixer} \mid \text{Explosion}) = \dfrac{P(\text{Ex} \mid \text{Elix})\, P(\text{Elix})}{P(\text{Ex} \mid \text{Elix}) P(\text{Elix}) + P(\text{Ex} \mid \text{Lead}) P(\text{Lead}) + P(\text{Ex} \mid \text{Phil})\, P(\text{Phil})}$

$$= \frac{.5 \times .4}{.5 \times .4 + 1 \times .2 + .8 \times .4}$$

$$= \frac{.2}{.72} = \frac{5}{18}$$

5) a) $\sum_{k=0}^{\infty} \frac{1}{3} c^k = \frac{1}{3} \frac{1}{1-c}$ $\leftarrow$ this should be 1

$= 1$ if $c = \frac{2}{3}$

b) $\frac{1}{3}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^4 + \frac{1}{3}\left(\frac{2}{3}\right)^5$

$= \dfrac{108 + 72 + 48 + 32}{3^6} = \dfrac{260}{729}$

6) a) $P(A \text{ but not } B) = P(A) - P(AB)$
$= x - z$

b) $P(\text{Neither } A \text{ nor } B) = 1 - P(A \cup B)$
$= 1 - P(A) - P(B) + P(AB)$
$= 1 - x - y + z$

7) WITH REPLACEMENT:

a) $\left(\frac{9}{20}\right)^6$

ways of choosing location of 3 reds among 6.

Choose 2 whites among remaining 3 spots.

b) $\binom{6}{3}\binom{3}{2}\left(\frac{9}{20}\right)^3\left(\frac{8}{20}\right)^2\left(\frac{3}{20}\right)^1$

c) Complement of "at least one W, at least one B"

is "No W OR No B"

$P(\text{No } W) = \left(\frac{12}{20}\right)^6 = a$

$P(\text{No } B) = \left(\frac{17}{20}\right)^6 = b$

$P(\text{No } W, \text{No } B) = \left(\frac{9}{20}\right)^6 = c$

$P(\text{No } W \text{ OR No } B) = a + b - c$

$P(\text{at least } 1\ W \text{ and at least } 1\ B) =$

$1 - a - b + c$

WITHOUT REPLACEMENT

a) $\dfrac{9\times8\times7\times6\times5\times4}{20\times19\times18\times17\times16\times15}$

b) $\binom{6}{3}\binom{3}{2}\dfrac{\overbrace{9\times8\times7}^{R}\times\overbrace{8\times7}^{W}\times3}{20\times19\times18\times17\times16\times15}$

c) $P(\text{No } W) = \dfrac{12\times11\times10\times9\times8\times7}{20\times19\times18\times17\times16\times15} = a$

$P(\text{No } B) = \dfrac{17\times16\times15\times14\times13\times12}{20\times19\times18\times17\times16\times15} = b$

$P(\text{No } W, B) = \dfrac{9\times8\times7\times6\times5\times4}{20\times19\times18\times17\times16\times15} = c$

So $P(\text{at least } 1W, 1B) = 1 - a - b + c$

8) Joint mass of #H, #T :

| T \ H | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $p^3$ |
| 1 | 0 | 0 | $3p^2\times(1-p)$ | 0 |
| 2 | 0 | $3p\times(1-p)^2$ | 0 | 0 |
| 3 | $(1-p)^3$ | 0 | 0 | 0 |

$P(H=3, T=0) = p^3$

$P(H=2, T=1) = 3p^2(1-p)$

$P(H=1, T=2) = 3p(1-p)^2$

$P(H=0, T=3) = (1-p)^3$

Mass of $X = \#H - \#T$

$P(X=3) = p^3$
$P(X=1) = 3p^2(1-p)$
$P(X=-1) = 3p(1-p)^2$
$P(X=-3) = (1-p)^3$

$E(X) = 3p^3 + 3p^2(1-p) - 3p(1-p)^2 - 3(1-p)^3$
$E(X^2) = 9p^3 + 3p^2(1-p) + 3p(1-p)^2 + 9(1-p)^3$
$Var(X) = E(X^2) - [E(X)]^2$

(For a), $p = \frac{1}{2}$, $E(X) = 0$
$Var(X) = 3$

9) $X$ = Gain, $X = \begin{cases} 10 & \bar{w} \text{ prob } .5 \\ 20 & \bar{w} \text{ prob } \frac{1}{3} \\ 30 & \bar{w} \text{ prob } \frac{1}{6} \end{cases}$

$E(X) = \frac{10}{2} + \frac{20}{3} + \frac{30}{6} = 16\frac{2}{3}$
$E(X^2) = \frac{100}{2} + \frac{400}{3} + \frac{900}{6} = 333\frac{1}{3}$
$Var(X) = 333\frac{1}{6} - (16\frac{2}{3})^2 = 55.\dot{5}$

10) $X$ = Score of randomly chosen student

By T.,  $P(|X-120| \le t) \ge 1 - \frac{64}{t^2}$ $\to Var(X)$

Want $t$ s.t. $1 - \frac{64}{t^2} = .9$

ie $\frac{64}{t^2} = \frac{1}{10}$

ie $t^2 = 640$

ie $t = 25.3$