Math 30440, Spring 2009

Exam 2 solutions

March 22, 2009

For questions 1 – 3, suppose that historical data indicates that the daytime high temperature in South Bend on Christmas day is normally distributed with mean -5 degrees Celsius and standard deviation of 12 degrees Celsius.

1. What is the distribution of the daytime high temperature on Christmas day in degrees Fahrenheit? (To convert from degrees Celsius to degrees Fahrenheit, first multiply by 9/5 and then add 32. For example, 100 degrees Celsius converts to 212 degrees Fahrenheit.)

   Solution: Let C_{25} be the daytime high on Christmas day measured in degrees Celsius; $X_{25} = N(-5, (12)^2)$. Let F_{25} be the same thing measured in degrees Fahrenheit. The connection between C_{25} and F_{25} is that $F_{25} = (9/5)C_{25} + 32$. So

   $$E(F_{25}) = E((9/5)C_{25} + 32) = (9/5)E(C_{25}) = 23$$

   and

   $$Var(F_{25}) = Var((9/5)C_{25} + 32) = (9/5)^2 Var(C_{25}) = (9/5)^2(12)^2.$$

   (Notice that when we take out the $9/5$ in the variance calculation, it gets squared, and that the $+32$ has no effect on variance). So F_{25} is a normal distribution with mean 23 and standard deviation $(9/5) * 12 = 21.6$.

2. Calculate the probability that the daytime high will be above freezing (above 0 degrees Celsius) on Christmas day 2010.

   Solution:

   $$P(C_{25} > 0) = P\left(Z > \frac{0+5}{12}\right) = 1 - P(Z \leq .416...) = 1 - .6616... = .3384...$$

   (where Z is a standard normal).

3. Suppose the high on December 26 is normally distributed with mean -6 degrees Celsius and standard deviation of 16 degrees Celsius. What is the distribution of the average daytime high temperature on Christmas weekend 2010 (December 25th and 26th)?

   Solution: Let C_{26} be the daytime high on December 26 measured in degrees Celsius; $X_{26} = N(-6, (16)^2)$. The average day time high over the two days is $(C_{25} + C_{26})/2$, which is again a normal distribution, with mean $(-5 + -6)/2 = -5.5$ and variance $((12)^2 + (16)^2)/4 = 100$ so standard deviation 10. (Notice that when we divide the sum by 2, we divide the variance by $2^2 4$.)

4. Suppose that the local paper has an average of 3 subscribers cancel each week. If we presume that there are many subscribers and subscribers decide to cancel independently, approximate the probability that 4 subscribers cancel next week.

   Solution: We model the number of subscribers who cancel in a week using a Poisson random variable X with parameter $\lambda = 3$. So the probability that (exactly) 4 subscribers cancel next week is

$$P(X = 4) = \frac{3^4}{4!} e^{-3}.$$

5. Suppose you are running late for school and have a choice of either waiting for the bus which will arrive sometime between now and an hour from now or calling a taxi which will arrive sometime between 6 and 24 minutes from now. If the arrival times for the taxi and bus are independent and uniformly distributed what is the probability the taxi will arrive before the bus?

   Solution: The density for X, the arrival time of the bus, is $f(x) = 1/60$ for $0 \leq x \leq 60$ (and 0 otherwise) and the density for Y, the arrival time of the taxi, is $f(y) = 1/18$ for $6 \leq y \leq 24$ (and 0 otherwise), so the joint density is $f(x, y) = (1/60) * (1/18) = 1/1080$ for the box $0 \leq x \leq 60, 6 \leq y \leq 24$ (and 0 otherwise). To find $P(Y < X)$ (which is the event that the taxi arrives first) we need to integrate the joint density over that part of the plane where the y-coordinate is less than the x-coordinate (i.e., on the lower side of the line $x = y$). This intersects the box $0 \leq x \leq 60, 6 \leq y \leq 24$ in the quadrilateral with vertices $(6, 6), (60, 6), (60, 24)$ and $(24, 24)$. Since we're integrating a constant function, the answer is just the area multiplied by $1/1080$, which is $760/1080 = .75$.

6. Suppose that each person passing through airport security has a 1% chance of being pulled aside for additional screening. If $10,000$ people passed through security at Midway airport today and the TSA makes the selections for additional screening independently give an exact expression for the probability that over 150 people are screened. (You don't need to numerically evaluate the expression).

   Solution: We model the number of people pulled aside for additional screening using a Binomial random variable X with $n = 10,000$ and $p = .01$, so the probability that over 150 are pulled aside is

$$P(X > 150) = \sum_{k=151}^{10000} \binom{10000}{k} (.01)^k (.99)^{10000-k}.$$

7. Suppose the half-life of Ce-141 is 30 days (in other words, the probability that an individual atom of Ce-141 will decay sometime within the next 30 days is exactly $\frac{1}{2}$). If C is the time it takes for a single atom of Ce-141 to decay compute the distributions for C. (Assume that the lifetime of an individual atom is exponentially distributed).

   Solution: Let X be the lifetime of a Ce-141 atom. X has an exponential distribution with some parameter λ that we do not know. So

$$P(X \leq 30) = \int_0^{30} \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_{x=0}^{30} = 1 - e^{-30\lambda}.$$

   But also we are given that $P(X \leq 30) = .5$. So $1 - e^{-30\lambda} = .5$ or $\lambda = (\ln 2)/30$. This is enough to completely specify the distribution.

8. The lifetime of component A is an exponential random variable with $\lambda = 12$. The lifetime of component B is exponential with $\lambda = 8$. Give an expression for the probability that component A fails before component B. (You don't need to evaluate the expression).

   Solution: The lifetime of A is X with density $f(x) = 12e^{-12x}$ for $0 \leq x$ (and 0 otherwise) and the lifetime of B is Y with density $f(y) = 8e^{-8y}$ for $0 \leq y$ (and 0 otherwise), so the joint density is $f(x, y) = 96e^{-12x-8y}$ in the first quadrant $0 \leq x, 0 \leq y \leq$ (and 0 otherwise). To find $P(X < Y)$ (which is the event that A fails first) we need to integrate the joint density over that part of the plane where the x-coordinate is less than the y-coordinate, in the first quadrant. This is the infinite triangle with vertices $(0, 0)$, $(0, \infty)$ and (∞, ∞) (the upper triangle in the first quadrant), and the expression we are looking for is

$$\int_{x=0}^{\infty} \int_{y=x}^{\infty} 96e^{-12x-8y}\, dydx \left(\text{or } \int_{y=0}^{\infty} \int_{x=0}^{y} 96e^{-12x-8y}\, dxdy\right).$$