Math 30440: Probability and Statistics
Spring Semester 2009
Exam 1 — Full solutions

1. A full house in Poker is a set of five cards made up of a three of a kind and a pair (e.g., $5\heartsuit, 5\clubsuit, 5\diamondsuit, 7\spadesuit, 7\diamondsuit$).

1. How many different full houses are there? (Ignoring the order of the cards in the hand).
   Solution: 13 ways to choose face value of set of 3, $\binom{4}{3}$ ways to choose actual three cards, 12 ways to choose face value of pair, $\binom{4}{2}$ ways to choosectual pair. So $13 * \binom{4}{3} * 12 * \binom{4}{2} = 3744$ in all.
2. I play the following game with you: you pay me a dollar. I then deal you 5 cards from a regular deck (with all possible hands of 5 cards equally likely). If your 5 cards form a full house, then I give you \$1000; otherwise I give you nothing. If X is the amount that you win playing this game, calculate $E(X)$. (**Clarification**: If you win, $X = 999$ (\$1000 minus your \$1 entry fee); if you lose, $X = -1$)
   Solution: $P(\text{full house}) = 3744/\binom{52}{5} = .00144....$ So
   $$X = \begin{cases} 1000 - 1 = 999 & \text{with probability } .00144... \\ -1 & \text{with probability } .99855... \end{cases}$$
   $E(X) = 999 * .00144... - 1 * .99855... = .44....$
3. Would you be willing to play this game with me 200 times? Explain why or why not.
   Solution: This is a matter of opinion. On the one hand, yes, because I expect to win 44 cents on average each time I play; so over 200 plays, I expect to win \$88. *But*: if I only play 200 times, what is the probability that I lose all 200 times? It's $(.99855)^{200} = .748....$ So although my *expected* winning over 200 tries is \$88, 75% of the time in which I play 200 times, I will lose \$200. This seems like a risky proposition. (If 200 was replaced by 200,000, I would definitely play).

2. Three companies produce Digital Converter Boxes.

- Company A has market share 60% and 8% of its products are defective
- Company B has market share 30% and 6% of its products are defective
- Company C has market share 10% and 10% of its products are defective

1. I buy a Digital Converter Box. What's the probability that it is defective?
   Solution:
   $$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) = .08 * .6 + .06 * .3 + .1 * .1 = .076.$$

2. Given that it is defective, how likely was it to have been produced by Company C?

Solution:

$$P(C|D) = \frac{P(CD)}{P(D)} = \frac{P(D|C)P(C)}{P(D)} = \frac{.01}{.076} = .13...$$

3. Decide whether each of the following statements is true or false, and give a *short* explanation of why:

1. ____ If events E and G are independent, then knowing $P(E)$ and $P(G)$ allows one to compute $P(E \cup G)$.

   Solution: Yes, because if E and G are independent, $P(EG) = P(E)P(G)$ and so $P(E \cup G) = P(E) + P(G) - P(EG) = P(E) + P(G) - P(E)P(G)$.

2. ____ Mutually exclusive events are always independent.

   Solution: No, not always (in fact almost never), because if A and B are mutually exclusive then $P(AB) = 0$. This will only equal $P(A)P(B)$ if at least one of $P(A)$, $P(B)$ is zero; so this is the only circumstance under which A and B are independent.

3. ____ A function $f(x) = \begin{cases} c(1-x^2) & \text{if } |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$ could be a valid probability density function, with an appropriate choice of c.

   Solution: No. $1 - x^2$ is negative for values of x close to 2 and -2, and positive for values of x close to 0. So no matter whether c is positive or negative, the "density function" would have negative values, which is not allowed. The only other option is $c = 0$, but then $\int_{-\infty}^{\infty} f(x)\, dx = 0$ instead of 1.

4. ____ If random variables X and Y are independent, then $E(XY) = E(X)E(Y)$.

   Solution: Yes. If they are continuous, then $f(x, y) = f_X(x)f_Y(y)$ and so

   $$E(XY) = \int\int xyf(x,y)\, dA = \int xf_X(x)\, dx \int yf_X(y)\, dy = E(X)E(Y).$$

   It's similar if they are discrete.

4. A machine has three components: A, B and C. For the machine to work,

- **EITHER** both A and B have to work
- **OR** C has to work.

(**Clarification**: "or" is meant here in the usual sense; the machine will work if A, B and C all work.) The probabilities of A, B and C working are 0.8, 0.8 and 0.6, respectively. Assume that components work or fail independently of each other.

1. What is the probability that the machine is working?
   Solution: Good configurations:

A, B work, C doesn't	Probability $.8 * .8 * .4 = .256$
A, B, C work	Probability $.8 * .8 * .6 = .384$
A, C work, B doesn't	Probability $.8 * .6 * .2 = .096$
B, C work, C doesn't	Probability $.8 * .6 * .2 = .096$
C works, A, B don't	Probability $.6 * .2 * .2 = .024$

   Total probability of working: $.256 + .384 + .096 + .096 + .024 = .856$

2. Given that the machine is currently working, what is the probability that component A is working?
   Solution: $P(\text{Working AND A working}) = .256 + .384 + .096 = .736$, so

$$P(\text{A working}|\text{working}) = \frac{.736}{.856} = .8598...$$

5. The probability density function of the random variable X is

$$f_X(x) = \begin{cases} (1 - |x|) & |x| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

1. Find the cumulative distribution function of X.
   Solution: X can never take values below -1, so $F_X(a) = 0$ for $a < -1$. It always takes values below 1, so $F_X(a) = 1$ for $a > 1$. For $-1 \leq a \leq 1$,

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\, dx = \int_{-1}^{a} (1 - |x|)\, dx$$

   If $a \leq 0$, the function we are integrating is $1 + x$, which has antiderivative $x + \frac{x^2}{2}$ and so the value of the integral is $a + \frac{a^2}{2} - \left(-1 + \frac{(-1)^2}{2}\right) = \frac{a^2}{2} + a + \frac{1}{2}$. If $a \geq 0$, then the integral is

$$\int_{-1}^{0} (1 + x)\, dx + \int_{0}^{a} (1 - x)\, dx = \frac{1}{2} + \left[x - \frac{x^2}{2}\right]_0^a = \frac{1}{2} + a - \frac{a^2}{2}.$$

   So in summary

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2} & \text{if } -1 \leq x \leq 0 \\ \frac{1}{2} + x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

2. Calculate the probability that $|X| > \frac{1}{2}$.
   Solution: $P(|X| > 1/2) = P(X > 1/2) + P(X < -1/2) = (1 - F_X(1/2)) + F_X(-1/2) = (1 - 7/8) + 1/8 = 1/4$.

6. A joint probability density function of random variables X and Y is given by the formula

$$f(x,y) = \begin{cases} 24xy & \text{if } x \geq 0, \ y \geq 0 \quad \text{and} \quad x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the cumulative distribution function $F_X(x)$ for the random variable X.
   Solution: $F_X(a)$ is 0 for $a < 0$ and 1 for $a > 1$. For $0 \leq a \leq 1$,

$$\begin{aligned} F_X(a) &= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(x,y) \, dydx \\ &= \int_{0}^{a} \int_{0}^{1-x} 24xy \, dydx \\ &= \int_{0}^{a} \left[12xy^2\right]_0^{1-x} \, dx \\ &= \int_{0}^{a} (12x - 24x^2 + 12x^3) \, dx \\ &= \left[6x^2 - 8x^3 + 3x^4\right]_0^a \\ &= 6a^2 - 8a^3 + 3a^4. \end{aligned}$$

2. Find $P(X \leq Y)$
   Solution:

$$P(X \leq Y) = \int_{0}^{1/2} \int_{x}^{1-x} 24xy \, dydx = 1/2.$$

3. Are X and Y independent?
   Solution: No. E.g., the information that X takes values close to 1 gives the information that Y takes values close to 0.