Answer Key for Exam $\boxed{\text{A}}$

1. Suppose the shoe size of men in South Bend is normally distributed with mean 10 and standard deviation of 2. What percentage of the population has a show size larger than 13?

Let $S \backsim \mathcal{N}(10, 2^2)$. Thus $\frac{S-10}{2} \backsim \mathcal{N}(0,1)$.

$$\mathrm{P}(S > 13) = \mathrm{P}((S-10) > 3) = \mathrm{P}\left(\frac{S-10}{2} > 1.5\right) \approx .067 = 6.7\%$$

Note we are assuming that shoe sizes are continuous, i.e., you can have a show size of 13.152...

2. If 15 customers purchase shoes from a particular South Bend shoe store today what is the probability that exactly 5 of them have a shoe size larger than 13?

$$\text{Let } X_i = \begin{cases} 1 & \text{if customer } i \text{ has shoe size} > 13 \\ 0 & \text{otherwise} \end{cases}$$

Now by above $\mathrm{P}(X_i = 1) = .067$ thus if $Y = \sum_{i=1}^{15} X_i$ then Y is binomial with $n = 15$ and $p = .067$.

$$\mathrm{P}(Y = 5) = \binom{15}{5}(.067)^5(1-.067)^{10} \approx .002$$

3. Kyle and Gordan have a contest to determine who has the lost the most weight over the course of a week. Using their own scales they each weigh themselves at the start and end of the week to determine how much weight they've lost and then compare these values to determine a winner. Suppose the error (measured weight − true weight) of Kyle's scale is $\backsim \mathcal{N}(2\,\text{lbs}, 3\,\text{lbs})$ and that of Gordon's is $\backsim \mathcal{N}(-3\,\text{lbs}, 4\,\text{lbs})$ and the scales declare Gordan is the winner by 2.5 lbs. What is the probability Gordon actually won?

> Let $K_S \backsim \mathcal{N}(2\,\text{lbs}, 3\,\text{lbs})$ be the error in Kyle's weight at the start and $K_F \backsim \mathcal{N}(2\,\text{lbs}, 3\,\text{lbs})$ be the error in Kyle's weight at the end. Let $K = K_F - K_S$ be the error in the amount of weight Kyle lost. Then $K \backsim \mathcal{N}(2-2, 3+3) = \mathcal{N}(0,6)$ and similarly if G is the error in the amount of weight Gordon lost $G \backsim \mathcal{N}(0,8)$. Thus $G - K \backsim \mathcal{N}(0, 6+8)$ so $\frac{G-K}{\sqrt{14}} \backsim \mathcal{N}(0,1)$. Gordon is the winner only if $2.5 + (G-K) > 0$.
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> $$\mathrm{P}\left(G - K > -2.5\right) = \mathrm{P}\left(\frac{G-K}{\sqrt{14}} > -.668\right) \approx .748$$

4. If the x and y components of the final position of a billiard ball hit hard on a $100'' \times 50''$ pool table have independent uniform distributions compute the probability that such a ball comes to rest within $20''$ of the center of the table.

> Note that since the x and y distributions are independent and uniform the probability the ball will land in any region is just proportional to the area of that region.
> $\frac{\pi 20^2}{50 \cdot 100} \approx .25$

5. Suppose that there are an average of 50 house fires in December in Indiana. If we assume that in december houses catch fire or not independently write an expression for the probability that exactly 15 houses catch fire next December?

> If F is the number of fires in December then F has a Poisson distribution with $\lambda = 50$.
> Hence $\mathrm{P}(F = 15) = e^{-50} \cdot \frac{50^{15}}{15!}$.

6. Suppose that Charlie's dog Fido runs off into the forest and painful past experience has taught Charlie that the time until his dog is sprayed by a skunk is exponentially distributed with an average of 1 hour. If the time it takes Charlie to race back home and return with Fido's favorite toy ranges between 15 and 30 minutes and is uniformly distributed over that interval give an expression for the probability that Charlie can lure Fido back with his favorite toy before he gets sprayed by a skunk.

> Let S be the time that Fido would get sprayed by the skunk and L be the time it takes to lure Fido back. We want to calculate $\mathrm{P}(L < S)$. $\mathrm{P}(L < S) = \mathrm{P}\left(S > \frac{1}{2}\right) + \mathrm{P}\left(L < S, S \leq \frac{1}{2}\right)$.
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> $$\mathrm{P}(S > 45) = \int_{.5}^{\infty} e^{-s}\mathrm{d}s$$
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> $$\mathrm{P}\left(L < S, S \leq \frac{1}{2}\right) = \int_{.25}^{.5}\int_{l}^{.5} 4 \cdot e^{-s}\mathrm{d}s\mathrm{d}l$$
>
> $$\mathrm{P}(L < S) = \int_{.5}^{\infty} e^{-s}\mathrm{d}s + \int_{.25}^{.5}\int_{l}^{.5} 4 \cdot e^{-s}\mathrm{d}s\mathrm{d}l$$