Math 30440, Spring 2009

Homework solutions

6.1) a) Possible values for $\frac{X_1+X_2}{2}$:

$\boxed{n=2}$

0 $\bar{w}$ prob .04

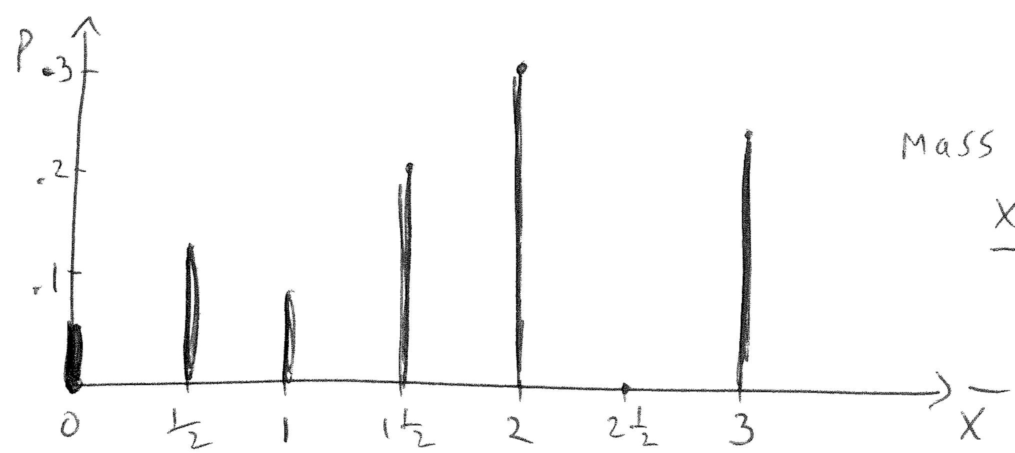
.5 $\bar{w}$ prob $2\times.06 = .12$

1 $\bar{w}$ prob .09

1.5 $\bar{w}$ prob $2\times.1 = .2$

2 $\bar{w}$ prob $2\times.15 = .3$

3 $\bar{w}$ prob .25



Mass function of $\frac{X_1+X_2}{2}$

$$E(\bar{X}) = \tfrac{1}{2}\left(E(X_1)+E(X_2)\right) = 1.8$$

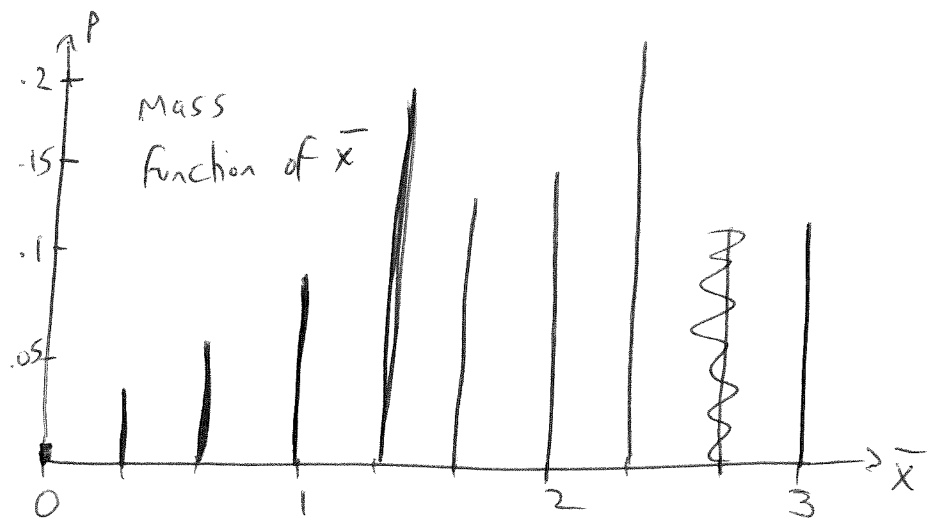
$$Var(\bar{X}) = \tfrac{1}{4}\left(Var(X_1)+Var(X_2)\right) = .78$$

b)

$\boxed{n = 3}$

X_1	X_2	X_3	$\bar{X}$	prob
0	0	0	0	.008
0	0	1	$\frac{1}{3}$	.012
0	0	3	1	.02
0	1	0	$\frac{1}{3}$	.012
0	1	1	$\frac{2}{3}$	.018
0	1	3	$\frac{4}{3}$	.03
0	3	0	1	.02
0	3	1	$\frac{4}{3}$	.03
0	3	3	2	.05
1	0	0	$\frac{1}{3}$	.012
1	0	1	$\frac{2}{3}$	.018
1	0	3	$\frac{4}{3}$	.03
1	1	0	$\frac{2}{3}$	.018
1	1	1	1	.027
1	1	3	$\frac{5}{3}$	.045
1	3	0	$\frac{4}{3}$	.03
1	3	1	$\frac{5}{3}$	.045
1	3	3	$\frac{7}{3}$	.075
3	0	0	1	.02
3	0	1	$\frac{4}{3}$	.03
3	0	3	2	.05
3	1	0	$\frac{4}{3}$	.03
3	1	1	$\frac{5}{3}$	.045
3	1	3	$\frac{7}{3}$	.075
3	3	0	2	.05
3	3	1	$\frac{7}{3}$	.075
3	3	3	3	.125

$\bar{X} =$	$p =$
0	.008
$\frac{1}{3}$	.036
$\frac{2}{3}$	.054
1	.087
$\frac{4}{3}$	.18
$\frac{5}{3}$	.135
2	.15
$\frac{7}{3}$	.225
3	.125



$$E(\bar{X}) = E(X_i) = 1.8$$

$$Var(\bar{X}) = \frac{Var(X_i)}{3} = .52$$

6.4) a) Each play:

Win 35 $\bar{w}$ prob $\frac{1}{38}$

Lose 1 $\bar{w}$ prob $\frac{37}{38}$

W = Win per game

$$E(W) = \frac{35}{38} - \frac{37}{38} = -\frac{1}{19} = -.0526\ldots$$

$$Var(W) = \left(35+\frac{1}{19}\right)^2 \frac{1}{38} + \left(\frac{18}{19}\right)^2 \frac{37}{38} \approx 33.2$$

You are down after 34 bets only if you lose all 34, probability $\left(\frac{37}{38}\right)^{34} \approx .4$

So up after 34 $\bar{w}$ probability $\approx .6$

b) $W_T = W_1 + \cdots + W_{1000}$ winnings after 1000 bets

$$E(W_T) = 1000\, E(W) = -52.6$$

$$Var(W_T) = 1000\, Var(W) = 33200$$

So $W_T \approx N(-52.6, 33200)$

$$P(W_T > 0) = P\left(\frac{W_T + 52.6}{\sqrt{33200}} > \frac{0 + 52.6}{\sqrt{33200}}\right)$$

$$= P(Z \geq .288)$$

$$\approx .39$$

c) If W_T = Winnings after 100 000 bets,

$W_T \approx N(-5260, 3.32\text{million})$

$P(W_T > 0) = P(Z > 2.88)$

$\approx .002$

6.6) Individual error : Uniform $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Sum of 50 errors : Mean $0 \times 50 = 0$

Variance $\frac{1}{2} \times 50 = 4.166\ldots$

(Variance of uniform is $\frac{1}{12}$)

So Sum of errors $\approx N(0, 4.166)$

$P(|Sum| > 3) = P(Sum > 3 \text{ or } Sum < -3)$

$= 2P(Sum > 3)$ (Sum symmetric around zero)

$= 2P\left(Z > \frac{3}{\sqrt{4.166}} = 1.469\right)$

$= .141$

6.7) For single roll X, $E(X) = 3.5$

$Var(X) = 2.9166\ldots$

Sum of 140 rolls has mean 490

variance 408.33...

So by CLT is $\approx N(490, 408.33)$

Requiring more than 140 rolls to get to 400

$=$

Being less than 400 after 140 rolls

$=$

$P(N(490, 408.33) < 400)$

$=$

$P(Z < -4.45) \approx 0$

6.9) X_i = lifetime of i^{th} part

$E(X_i) = 100$, $Var(X_i) = (20)^2$

$\bar{X}$ = Average of 16

$E(\bar{X}) = 100$, $Var(\bar{X}) = \frac{(20)^2}{16} = 25$

So $\bar{X} \approx N(100, 25)$

a) $P(\bar{X} < 104) = P(Z < .8)$

$= .7881$

b) $P(98 \le \bar{X} \le 104) = P\left(\frac{98-100}{5} \le \frac{\bar{X}-100}{5} \le \frac{104-100}{5}\right)$

$= P(-.4 \le Z \le .8)$

$= \Phi(.8) - \Phi(-.4)$

$= .443$

6.12) a) Average for group of 25 is approximately $N\left(77, \frac{(15)^2}{25}\right) = N(77, 9)$

$$P(72 \le N(77,9) \le 82)$$

$$= P\left(\frac{-5}{3} \le Z \le \frac{5}{3}\right) \approx .9044$$

b) Average for group of 64 is approx $N\left(77, \frac{(15)^2}{64}\right) = N(77, 3.51)$

$$P(72 \le N(77, 3.51) \le 82)$$

$$= P\left(\frac{72-77}{\sqrt{3.51}} \le N(0,1) \le \frac{82-77}{\sqrt{3.51}}\right)$$

$$= P(-2.66\ldots \le Z \le 2.66\ldots) = .9924$$

c) $P(\text{Average for } 25 > \text{Average for } 64)$

$$= P(N(77,9) > N(77, 3.51))$$

$$= P(N(0, 12.51) > 0) \approx .5 \text{ (by symmetry)}$$

d) Class of size 25, since it has greater variance in its average

6.15) a) X is <u>not</u> binomial; it is # of successes in repeated trials, but the success probabilities are not constant

b) $X_A = \text{Binomial}(32, .5)$
$\approx N(16, 8)$
$X_B = \text{Binomial}(28, .7)$
$\approx N(19.6, 5.88)$

c) $X = X_A + X_B$

d) $X \approx N(16 + 19.6, 8 + 5.88)$
$= N(35.6, 13.88)$

$$P(X > 40) = P\left(\frac{X - 35.6}{\sqrt{13.88}} > \frac{40 - 35.6}{\sqrt{13.88}}\right)$$
$$= P(Z > 1.18\ldots)$$
$$= .119$$

6.22) For sample of size n, let X_n = number in favour

$$X_n = \text{Bin}(n, .52)$$

$$\approx \text{Normal}(.52n, n(.52)(.48))$$

$$= N(.52n, .2496n)$$

$$P(\text{Proportion} \geq 50\%) = P(X_n \geq \tfrac{n}{2})$$

$$= P(N(.52n, .249n) \geq .5n)$$

$$= P\left(Z \geq \frac{.5n - .52n}{\sqrt{.2496n}}\right)$$

$$= P(Z \geq -.040032\sqrt{n})$$

a) $n = 10$, $P(Z \geq -.1265) \approx$ ~~.6711~~ $.5517$

(exact answer: $.6711$)

b) $n = 100$, $P(Z \geq -.4) = .6554$

(exact answer $.6918$)

c) $n = 1000$, $P(Z \geq -1.26) \approx .8962$

(exact answer $.9027$)

d) $n = 10{,}000$, $P(Z \geq 4) \approx 1$.

6.30) Let n components be in stock.

Expected lifetime : $100n$

Variance : $(30)^2 n$

Distribution of lifetime : $\approx N(100n, (30)^2 n)$

Want $P(N(100n, (30)^2 n) \geq 2000) = .95$

ie $P\left(Z \geq \frac{2000 - 100n}{(30)\sqrt{n}}\right) = .95$

But $P(Z \geq -1.645) = .95$,

So need $\frac{2000 - 100n}{30\sqrt{n}} = -1.645$

Should take $n = 23$