Math 30440, Homework 5 Solutions

All problems from Ross, Chapter 5

2) Assume that errors on different bits are independent.

P(message will be incorrectly decoded)

= P(3, 4 or 5 errors)

Let X = # errors.

X is Binomial $(5, .2)$

$$P(X = 3, 4 \text{ or } 5) = \sum_{k=3}^{5} \binom{5}{k} .2^k .8^{5-k}$$

$$\boxed{= .0579}$$

4) For each child, 4 possibilities for gene pair:

$$\left.\begin{array}{l} rr \\ rd \\ dr \\ dd \end{array}\right\} \text{ each with probability } \frac{1}{4}.$$

All of these except the first lead to dominant appearance.

So $P(\text{Child has dominant appearance}) = \frac{3}{4}$.

Let $X = \#$ children with dominant appearance

$X = \text{Binomial}\left(4, \frac{3}{4}\right)$

$P(X = 3) = \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \frac{27}{64}$

8) a) $\dfrac{P(X=k+1)}{P(X=k)} = \dfrac{\binom{n}{k+1} p^{k+1} (1-p)^{n-(k+1)}}{\binom{n}{k} p^k (1-p)^{n-k}}$

$= \dfrac{\cancel{n!}\; k!\,(n-k)!}{(k+1)!\,(n-(k+1))!\;\cancel{n!}} \dfrac{p}{1-p}$

$= \dfrac{n-k}{k+1} \dfrac{p}{1-p}$, as claimed.

b) As long as $\dfrac{P(X=k+1)}{P(X=k)} > 1$, sequence is increasing; when $\dfrac{P(X=k+1)}{P(X=k)}$ is < 1, sequence is decreasing.

$\dfrac{n-k}{k+1} \dfrac{p}{1-p} = 1 \iff (n-k)p = (1-p)(k+1)$

$\iff np = k+1-p$

$\iff (n+1)p = k+1$

$\iff k = (n+1)p - 1$

So sequence changes from increasing to decreasing around $k \approx (n+1)p$

9) $$E(e^{tX}) = \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}(pe^t)^k(1-p)^{n-k}$$

$$= \boxed{(pe^t + (1-p))^n} \quad \text{(Binomial Theorem)}$$

$$= \phi(t)$$

$$\phi'(t) = n(pe^t + 1 - p)^{n-1}pe^t$$

$$\phi'(0) = \boxed{np = E(X)}$$

$$\phi''(t) = n(n-1)(pe^t + 1 - p)^{n-2}p^2e^{2t} + np(pe^t + 1 - p)^{n-1}e^t$$

$$\phi''(0) = n(n-1)p^2 + np = E(X^2)$$

Product rule

So $$Var(X) = E(X^2) - (E(X))^2$$

$$= n(n-1)p^2 - n^2p^2 + np$$

$$= np - np^2$$

$$= \boxed{np(1-p)}$$

10) a) Approx .1839... Exact .1938...

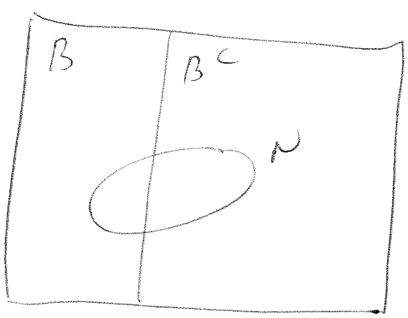
$\uparrow$ $np = 1$, so approx is $\frac{1^2}{2!}e^{-1}$

$\uparrow$ $\binom{10}{2} .1^2 .9^8$

b) Approx .3678... Exact .3487

c) Approx .0723... Exact .0660

12) This is a Bayes' Formula question



$B = \{\text{Medicine effective}\}$

$P(B) = .75$

$P(B^c) = .25$

$N = \{\text{No Colds}\}$

$$P(N|B) = P(\text{Poisson}(2) = 0) = e^{-2}$$
$$P(N|B^c) = P(\text{Poisson}(3) = 0) = e^{-3}$$

$$P(B|N) = \frac{P(N|B)P(B)}{P(N|B)P(B) + P(N|B^c)P(B^c)}$$

$$= \frac{e^{-2} \cdot .75}{e^{-2} \cdot .75 + e^{-3} \cdot .25} = \boxed{.8908\ldots}$$

14) Assuming birthdays are equally likely to fall on any one of 365 days (ignoring Feb 29), and that different people's birthdays are independent of each other:

a) For particular couple, $P(\text{both born on Apr. 30}) = \left(\frac{1}{365}\right)^2$

So # couples with both born on April 30 is Binomial with $n = 80{,}000$, $p = \left(\frac{1}{365}\right)^2$.

This is $\approx$ Poisson with $\lambda = \frac{80000}{(365)^2}$

So $P(0 \text{ couples}) \approx e^{\frac{-80000}{(365)^2}}$,

So $P(\text{at least 1 couple}) \approx 1 - e^{\frac{-80000}{(365)^2}}$

$= .45\ldots$

b) Now $p = 365$,

So $P(\text{at least one couple}) \approx 1 - e^{\frac{-80000}{365}}$

$= 1 \; (-10^{-96})$

17) $P(X=k) = \dfrac{\lambda^k}{k!} e^{-\lambda}$

$P(X=k+1) = \dfrac{\lambda^{k+1}}{(k+1)!} e^{-\lambda}$

So $\dfrac{P(X=k)}{P(X=k+1)} = \dfrac{k+1}{\lambda}$

For k (approximately) less than λ, the ratio is less than 1, so mass function is <u>increasing</u>;

for k greater than λ, it is greater than 1, so mass function is <u>decreasing</u>.

(See also question 8)

18) X = number of working components from sample of size 10

X is hypergeometric with $\underset{\text{good components}}{N=80}$, $\underset{\text{defective}}{M=20}$, $\underset{\text{sample size}}{n=10}$

$$P(X=9 \text{ or } 10) = \frac{\binom{80}{9}\binom{20}{1}}{\binom{100}{10}} + \frac{\binom{80}{10}\binom{20}{0}}{\binom{100}{10}}$$

$$\boxed{= .3630\ldots}$$

20) a) $P(X=k) = (1-p)^{k-1} p \quad , \quad k = 1, 2, \ldots$

probability of $k-1$ failures

probability of 1 success

b) Easiest to compute moment generating function:

$$\phi_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} (e^t(1-p))^k$$

$$= \frac{p e^t (1-p)}{(1-p)(1-e^t(1-p))} \quad \left(\text{sum of geometric series}\right)$$

$$= \frac{pe^t}{1-e^t(1-p)}$$

$$\phi_X'(t) = \frac{(1-e^t(1-p))pe^t - pe^t(-e^t(1-p))}{(1-e^t(1-p))^2}$$

$$= \frac{pe^t}{(1-e^t(1-p))^2}$$

$\phi_X'(0) = \frac{p}{p^2} = \frac{1}{p}$, so $\boxed{E(X) = \frac{1}{p}}$

c) $\boxed{P(Y=k) = (1-p)^{k-r} p^r \binom{k-1}{r-1}}$, $k = r, r+1, \ldots$

r^{th} success must be at end; first $r-1$ successes can be put anywhere among the first $k-1$ trials

d) $Y = Y_1 + \ldots + Y_r$, where Y_i is number of trials between $(i-1)^{st}$ success and i^{th} success.

Y_i is Geometric with parameter p,

So $E(Y) = E(Y_1) + \ldots + E(Y_r)$

$= \frac{1}{p} + \ldots + \frac{1}{p}$ (r times)

$= \boxed{\frac{r}{p}}$

21) $U = \text{Uniform}(0,1)$

Density $f(x) = \begin{cases} 0 & \text{if } x < 0,\ x > 1 \\ 1 & \text{if } 0 \le x \le 1 \end{cases}$

Distribution of $a + (b-a)U$: Clearly, it only takes values between a $(U=0)$ and b $(U=1)$

So $F = 0$ for $x \le a$
$\quad\ \ 1$ for $x > b$

For $a \le x \le b$,

$$F(x) = P(a + (b-a)U \le x)$$
$$= P\left(U \le \frac{x-a}{b-a}\right)$$
$$= \int_0^{\frac{x-a}{b-a}} 1 \, dt = \frac{x-a}{b-a}$$

Density of $a + (b-a)U$:

$$f(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases} \quad \left(\text{Derivative of } \frac{x-a}{b-a}\right)$$

This is exactly density of Uniform (a,b).

22) Let X = time bus arrives (measured in minutes from 10am)

X is Uniform $(0, 30)$, so has density

$$f(x) = \begin{cases} 0 & \text{if } x < 0,\ x > 30 \\ \frac{1}{30} & \text{if } 0 \le x \le 30 \end{cases}$$

P(Wait longer than 10 minutes)

$$= P(X > 10) = \boxed{\frac{2}{3}}$$

P(wait an additional 10 minutes, given that bus hasn't arrived at 10:15)

$$= P(X > 25 \mid X > 15) = \frac{P(X > 25 \text{ AND } X > 15)}{P(X > 15)}$$

$$= \frac{P(X > 25)}{P(X > 15)}$$

$$= \frac{1/6}{\frac{1}{2}} = \boxed{\frac{1}{3}}$$

(Note: second answer different from first; Uniform distribution is <u>not</u> memoryless!)