MATH 30440 — Homework 4 Solutions

(all problems from Ross, Chapter 4)

24) Say company charges x. Company's profit is

x with probability $1-p$

$x-A$ with probability p

$$E(\text{profit}) = x(1-p) + (x-A)p = x - Ap$$

<u>Want</u> $E(\text{profit}) = \frac{A}{10}$; so need x to satisfy

$$x - Ap = \frac{A}{10} \quad \text{or} \quad \boxed{x = Ap + \frac{A}{10}}$$

25) a) I expect $\boxed{E(X)}$ to be larger ... a randomly chosen student is more likely to have been on a large bus (there are more such students)

b) $E(X) = 40\left(\frac{40}{148}\right) + 33\left(\frac{33}{148}\right) + 25\left(\frac{33}{148}\right) + 50\left(\frac{50}{148}\right)$

$\approx \boxed{39.28}$

$E(Y) = 40\left(\frac{1}{4}\right) + 33\left(\frac{1}{4}\right) + 25\left(\frac{1}{4}\right) + 50\left(\frac{1}{4}\right) = \boxed{37}$

27) $\int f\,dx = 1$, so $\int_0^1 (a+bx^2)\,dx = 1$ (density integrates to 1)

ie $a + \frac{b}{3} = 1$

$\int xf\,dx = \frac{3}{5}$, so $\int_0^1 (ax + bx^3)\,dx = \frac{3}{5}$

ie $\frac{a}{2} + \frac{b}{4} = \frac{3}{5}$

Solving, $\boxed{a = \frac{3}{5},\ b = \frac{6}{5}}$

30) a) $P(X^n \le a) = \begin{cases} 0 & \text{if } a < 0 \\ 1 & \text{if } a > 1 \end{cases}$

For $0 \le a \le 1$,

$P(X^n \le a) = P(X \le a^{\frac{1}{n}}) = \int_0^{a^{\frac{1}{n}}} f(x)\,dx = a^{\frac{1}{n}}$

So $F_{X^n}(a) = \begin{cases} 0 & \text{if } a < 0 \\ a^{\frac{1}{n}} & \text{if } 0 \le a \le 1 \\ 1 & \text{if } a > 1 \end{cases}$

and $f_{X^n}(x) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x > 1 \\ \frac{1}{n} x^{\frac{1}{n}-1} & \text{if } 0 \le x \le 1 \end{cases}$

So $E(X^n) = \int_0^1 x \frac{1}{n} x^{\frac{1}{n}-1}\,dx = \frac{1}{n}\int_0^1 x^{\frac{1}{n}}\,dx = \boxed{\frac{1}{n+1}}$

b) $E(X^n) = \int_0^1 x^n f(x)\,dx = \int_0^1 x^n\,dx = \boxed{\frac{1}{n+1}}$

32) a) $E((2+4X)^2) = E(4+16X+16X^2)$
$= E(4) + 16E(X) + 16E(X^2)$
$= 4 + 16\times 2 + 16\times 8$
$= \boxed{164}$

b) $E(X^2+(X+1)^2) = E(X^2) + E(X^2) + 2E(X) + 1$
$= \boxed{21}$

34) a) Median m satisfies $F(m) = \frac{1}{2}$; so

$$\int_0^m e^{-x}\,dx = \frac{1}{2}.$$

This is $1 - e^{-m} = \frac{1}{2}$ or $\boxed{m = \ln 2 \approx .69}$

b) $\int_0^m 1\,dx = \frac{1}{2}$ same as $\boxed{m = \frac{1}{2}}$

39) a) $E(X) = \frac{1}{4}1 + \frac{1}{4}2 + \frac{1}{4}3 + \frac{1}{4}4 = \boxed{2\frac{1}{2}}$

b) $Var(X) = \frac{1}{4}(-1\frac{1}{2})^2 + \frac{1}{4}(-\frac{1}{2})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(1\frac{1}{2})^2$
$= \boxed{1\frac{1}{4}}$

40) $Var(X) = p_1(1-2)^2 + p_2(2-2)^2 + p_3(3-2)^2$

$= p_1 + p_3$

a) To maximize variance, take $p_2 = 0$ to get $Var(X) = p_1 + p_3 = 1$ (clearly the largest $p_1 + p_3$ can be).

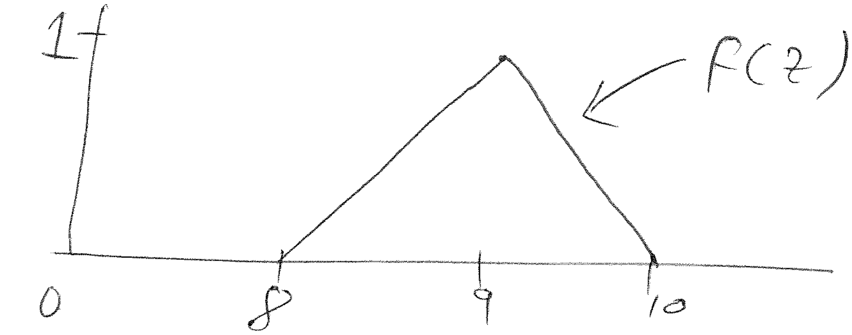
Since $E(X) = p_1 + 2p_2 + 3p_3 = 2$, will in this case also need to have $p_1 + 3p_3 = 2$

$\boxed{p_1 = \frac{1}{2},\ p_3 = \frac{1}{2}}$ (and $p_2 = 0$) solves

b) To minize variance, take $\boxed{p_2 = 1}$ so that $p_1 = p_3 = 0$, $Var(X) = 0$ (clearly the smallest it can be

(Works with expectation condition since $2p_2 = 2$ in this case)

43) a)



$$E(X) = \int z f(z)\,dz$$

$$= \boxed{\int_8^9 z(z-8)\,dz + \int_9^{10} z(10-z)\,dz}$$

$$= \ldots$$

$$E(X^2) = \int_8^9 z^2(z-8)\,dz + \int_9^{10} z^2(10-z)\,dz$$

$$= \ldots$$

$$Var(X) = \boxed{E(X^2) - [E(X)]^2}$$

b) $$\text{Profit}(z) = \begin{cases} -\left(\frac{z}{15} + .35\right) & \text{if } 8 \le z < 8.25 \\ 2 - \left(\frac{z}{15} + .35\right) & \text{if } 8.25 \le z \le 10 \end{cases}$$

(and 0 otherwise)

So $$E(\text{profit}) = \int \text{profit}(z) f(z)\,dz$$

$$\boxed{= \int_8^{8.25} -\left(\tfrac{z}{15} + .35\right)(z-8)\,dz + \int_{8.25}^{9} \left(2 - \left(\tfrac{z}{15} + .35\right)\right)(z-8)\,dz + \int_9^{10} \left(2 - \left(\tfrac{z}{15} + .35\right)\right)(10-z)\,dz} = \ldots$$

44) Note that "$0 \le u, v \le 1$" means "$0 \le u \le 1$"
AND
"$0 \le v \le 1$"

Notation $f_{XY}(u,v)$ indicates that variable u refers to X, v refers to Y.

a) $$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv$$

$$= \int_0^1 (u+v)\,dv = \boxed{u + \frac{1}{2}}$$

$$f_Y(v) = \boxed{v + \frac{1}{2}}$$

b) $$E(X) = \int_0^1 u\left(u+\tfrac{1}{2}\right)du = \boxed{\frac{7}{12}}$$

$$Var(X) = E\left((X - E(X))^2\right)$$

$$= \int_0^1 \left(u - \tfrac{7}{12}\right)^2 \left(u + \tfrac{1}{2}\right) du$$

$$= \boxed{\frac{11}{144}}$$

54) $E(e^{tx}) = \int_0^1 e^{tx} \cdot 1 \, dx$

$= \left[\frac{e^{tx}}{t} \right]_0^1$

$= \boxed{\frac{e^t - 1}{t}}$

To differentiate this, best to use power series:

$\frac{e^t - 1}{t} = \frac{(1 + t + \frac{t^2}{2!} + \dots) - 1}{t}$

$= \frac{t + \frac{t^2}{2!} + \dots}{t}$

$= 1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots$

n^{th} derivative at 0 is $\frac{1}{n+1}$, so

$\boxed{E(X^n) = \frac{1}{n+1}}$

(Check: see problem 30 of this set)

56) a) Let X be score. $E(X) = 75$.

By Markov, $P(X > 85) \boxed{\leq \frac{75}{85}} \approx .88$

b) $Var(X) = 25$

By Tchebychev,

$$P(|X - 75| > 10) \leq \frac{25}{(10)^2} = \frac{1}{4}$$

So $P(|X - 75| \leq 10) \geq \frac{3}{4}$

ie $P(65 \leq X \leq 85) \boxed{\geq \frac{3}{4}}$

c) Suppose n students take exam. By weak law of large numbers,

$$P\left(\left|\frac{X_1 + \cdots + X_n}{n} - 75\right| > 5\right) \leq \frac{25}{n(5)^2} = \frac{1}{n}$$

So $P\left(\text{Class average is within } 5 \text{ of } 75\right) \geq 1 - \frac{1}{n}$.

$\boxed{n = 10}$ enough to be certain that this is $\geq .9$