MATH 30440, SPRING 2009 Homework 3 Solns

1) $P(X=k)=0$ for $k=7,8,9,10$

(Can't have more than 5 people above top ranked woman)

For $k=1,2,3,4,5,6$,

$$P(X=k) = \frac{5\times4\times\ldots\times(5-(k-2))\times5\times(10-k)!}{10!}$$

($(5\times4\times\ldots\times(5-(k-2)))$ counts # ways of ordering the $k-1$ men above first woman; 5 counts # ways of choosing woman in k^{th} position; $(10-k)!$ counts # ways of ordering remaining $(10-k)$ people)

So:

$$P(X=1) = \frac{5\times9!}{10!} = \frac{1}{2}$$

$$P(X=2) = \frac{5\times5\times8!}{10!} = .2\dot{7}$$

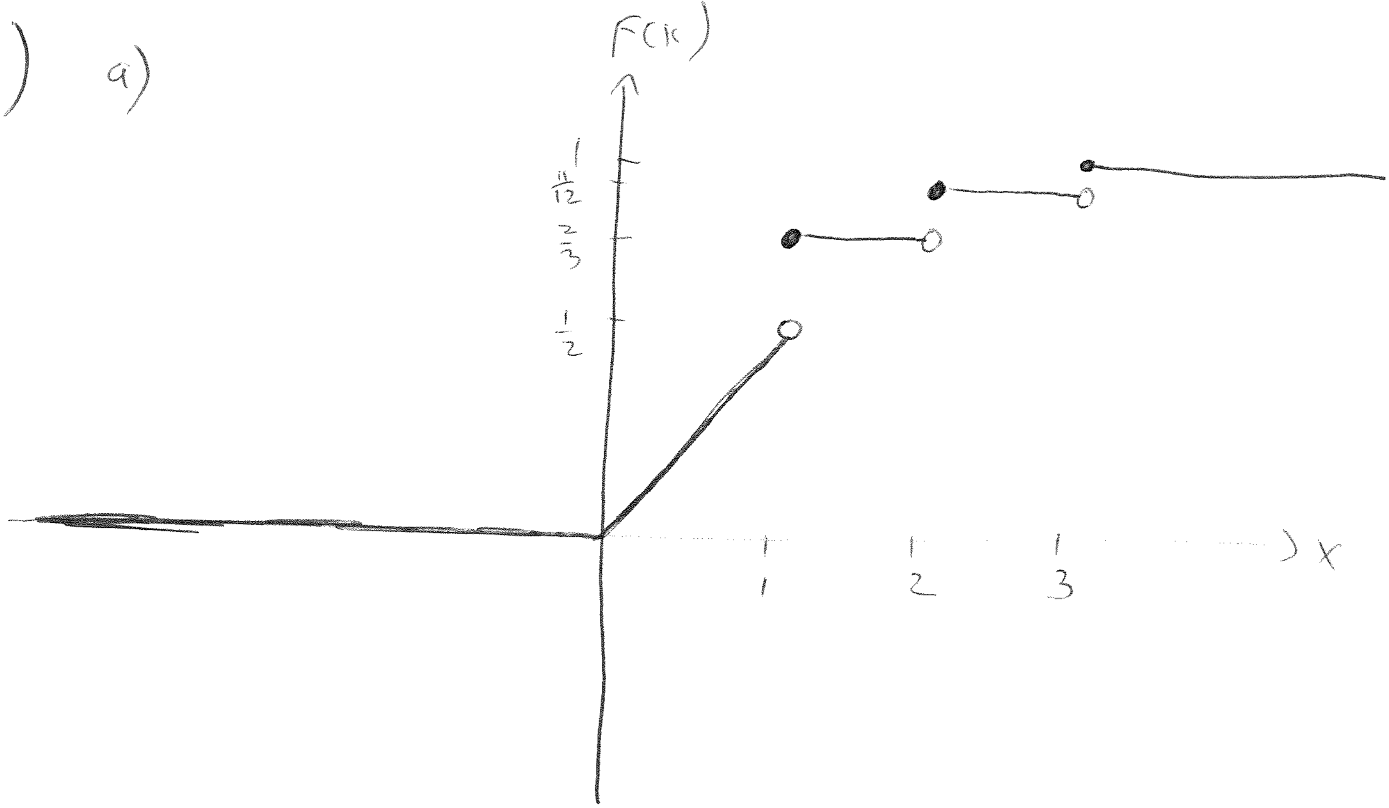
$$P(X=3) = \frac{5\times4\times5\times7!}{10!} = .13\dot{8}$$

$$P(X=4) = \quad = .059\ldots$$

$$P(X=5) = \quad = .0198\ldots$$

$$P(X=6) = \quad = .0039\ldots$$

4) a)



b) $P(X > \frac{1}{2}) = 1 - P(X \le \frac{1}{2})$

$= 1 - F(\frac{1}{2})$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

c) $P(2 < X \le 4) = F(4) - F(2)$

$= 1 - \frac{11}{12} = \frac{1}{12}$

d) $P(X < 3) = P(X \le 3) - P(X = 3)$

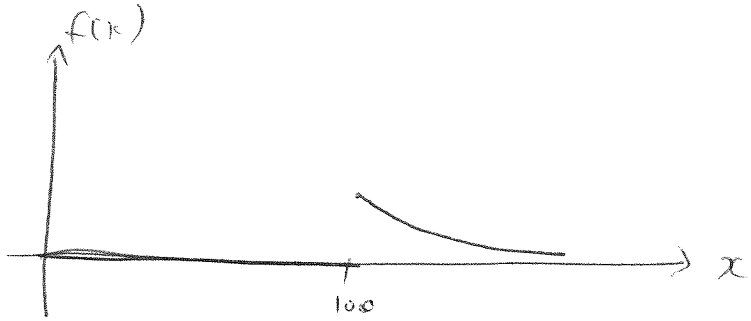
$= F(3) - \frac{1}{12}$

$= \frac{11}{12}$

F jumps by $\frac{1}{12}$ at 3, So $P(X=3)$ must be $\frac{1}{12}$

e) $P(X=1) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ (This is the amount by which F jumps going from below 1 to 1)

7)



P(i^{th} tube needs to be replaced within 150 hours)

$$= \int_{-\infty}^{150} f(x)\,dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= \left[\frac{-100}{x} \right]_{100}^{150} = \frac{1}{3}$$

P(a particular 2 from among 5 will need to be replaced within 150 hours, but not the other 3)

$$= \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3$$

$$P(\text{Exactly } 2) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$$

ways of choosing the 2

$$= \frac{80}{243}$$

8) To find c, need to solve $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{\infty} ce^{-2x}\,dx$$

$$= \left[-\frac{c}{2}e^{-2x}\right]_0^{\infty}$$

$$= \frac{c}{2}, \quad \text{so} \quad c = 2$$

$$P(X > 2) = \int_2^{\infty} 2e^{-2x}\,dx$$

$$= \left[-e^{-2x}\right]_2^{\infty} = e^{-4}$$

9) There are 10 equally likely orderings of the five transistors:

GGDDD $\longrightarrow$ $N_1 = 3$, $N_2 = 1$
GDGDD $\longrightarrow$ $N_1 = 2$, $N_2 = 2$
GDDGD $\longrightarrow$ $N_1 = 2$, $N_2 = 1$
GDDDG $\longrightarrow$ $N_1 = 2$, $N_2 = 1$
DGGDD $\longrightarrow$ $N_1 = 1$, $N_2 = 3$
DGDGD $\longrightarrow$ $N_1 = 1$, $N_2 = 2$
DGDDG $\longrightarrow$ $N_1 = 1$, $N_2 = 2$
DDGGD $\longrightarrow$ $N_1 = 1$, $N_2 = 1$
DDGDG $\longrightarrow$ $N_1 = 1$, $N_2 = 1$
DDDGG $\longrightarrow$ $N_1 = 1$, $N_2 = 1$

So here's the joint mass function

$p(1,1) = \frac{3}{10}$

$p(1,2) = \frac{2}{10}$

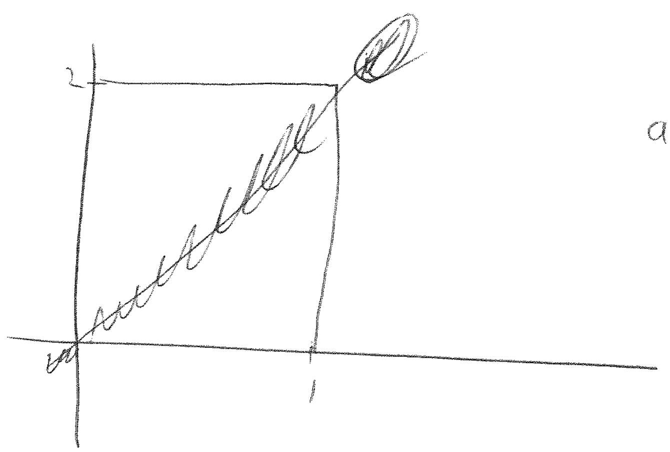
$p(1,3) = \frac{1}{10}$

$p(2,1) = \frac{2}{10}$

$p(2,2) = \frac{1}{10}$

$p(3,1) = \frac{1}{10}$

10)



a) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$

$= \int_0^2 \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$

$= \int_0^2 \left[\frac{6}{7}\left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]\right]_0^1 dy$

$= \int_0^2 \frac{6}{7}\left[\frac{1}{3} + \frac{y}{4}\right] dy$

$= \left[\frac{6}{7}\left[\frac{y}{3} + \frac{y^2}{8}\right]\right]_0^2$

$= 1$

So f is a joint density.

b) $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$

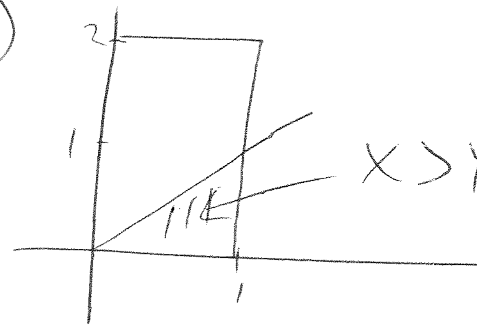
$= \int_0^2 \frac{6}{7}\left[x^2 + \frac{xy}{2}\right] dy$

$= \left[\frac{6}{7}\left[x^2 y + \frac{xy^2}{4}\right]\right]_0^2$

$= \frac{6}{7}\left[2x^2 + x\right] \longrightarrow$ For $0 \le x < 1$;

otherwise, $f_X(x) = 0$

c)



$X > Y$ exactly inside this region

$P(X>Y) = \iint_{\text{shaded region}} f(x,y)\,dA = \int_{x=0}^{1} \int_{y=0}^{x} \frac{6}{7}\left[x^2 + \frac{xy}{2}\right] dy\,dx$

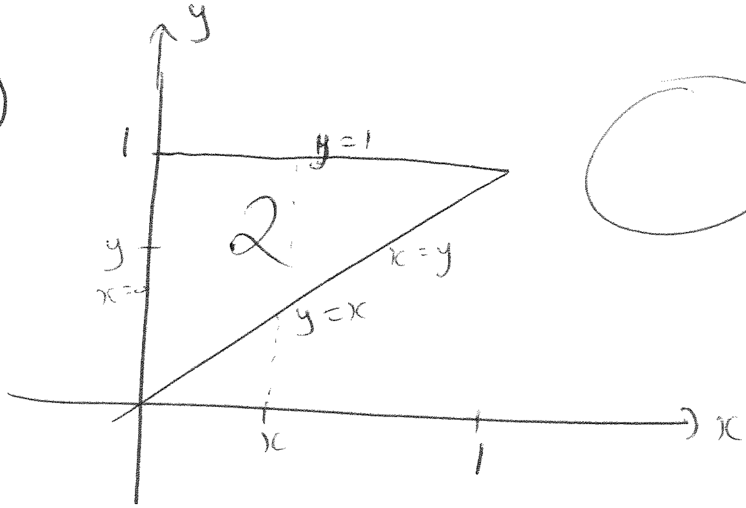
$= \int_{x=0}^{1} \left[\frac{6}{7}\left[x^2 y + \frac{xy^2}{4}\right]\right]_{y=0}^{x} dx$

$= \int_0^1 \frac{6}{7}\left[x^3 + \frac{x^3}{4}\right] dx$

$= \frac{15}{14}\int_0^1 x^3\,dx$

$= \cancel{\frac{30}{14}}\ \frac{15}{56}$

13)



a) $f_X(x) = \begin{cases} 0 \text{ if } x < 0 \text{ or } x > 1 \\ \int_x^1 2\,dy = [2y]_x^1 = 2 - 2x, & 0 \le x \le 1 \end{cases}$

b) $f_Y(y) = \begin{cases} 0 \text{ if } y < 0 \text{ or } y > 1 \\ \int_0^y 2\,dx = [2x]_0^y = 2y, & 0 \le y \le 1 \end{cases}$

c) No : Knowing something about X (eg, that X is close to 1) gives information about Y (eg, that Y is close to 1)

Also : $f_X(x) f_Y(y) = \begin{cases} 2y(2-2x) & 0 \le x, y \le 1 \\ 0 & \text{otherwise}, \end{cases}$

So $f(x,y) \neq f_X(x) f_Y(y)$

14) let A be event $a \le X \le b$
B " " $c \le Y \le d$

$$P(X \in A, Y \in B) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \int_a^b \int_c^d h(x)\,\ell(y)\,dy\,dx$$
$$= \left[\int_a^b h(x)\,dx\right]\left[\int_c^d \ell(y)\,dy\right]$$

If this last line is $P(X \in A)P(Y \in B)$, then ~~indeed~~ X and Y are independent.

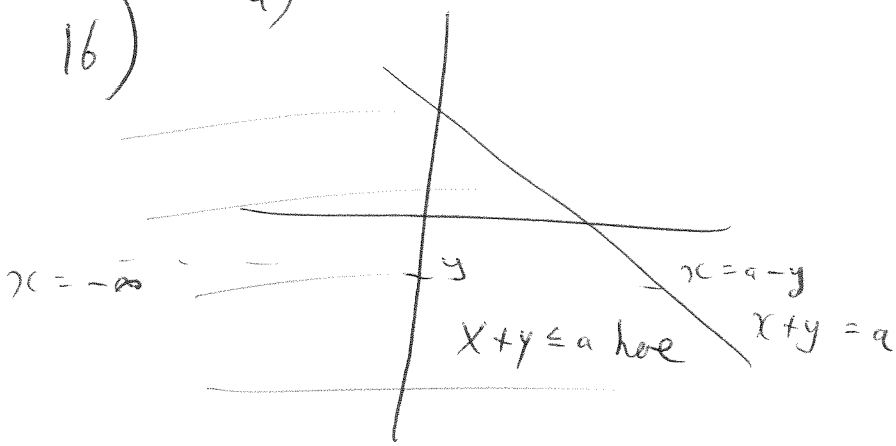
well, $P(X \in A) = \int_a^b \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$
$$= \int_a^b h(x)\,dx \int_{-\infty}^{+\infty} \ell(y)\,dy$$

and $P(\cancel{X} \in B) = \int_{-\infty}^{+\infty} \int_c^d f(x,y)\,dy\,dx$
$$= \int_{-\infty}^{+\infty} h(x)\,dx \int_c^d \ell(y)\,dy.$$

So $P(\cancel{X} \in A)P(Y \in B) = \int_a^b h(x)\,dx \int_c^d \ell(y)\,dy \left[\int_{-\infty}^{+\infty} h(x)\,dy\right]\left[\int_{-\infty}^{+\infty} \ell(y)\,dy\right]$
$$= \int_a^b h\,dx \int_c^d \ell\,dy \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy}_{=1,\ \text{since } f \text{ is a joint density.}}$$

So $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$, and they are independent.

16) a)



$$P(X+Y\le a) = \iint_{\text{Shaded region}} f(x,y)\,dA$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{a-y} f(x,y)\,dx\,dy$$

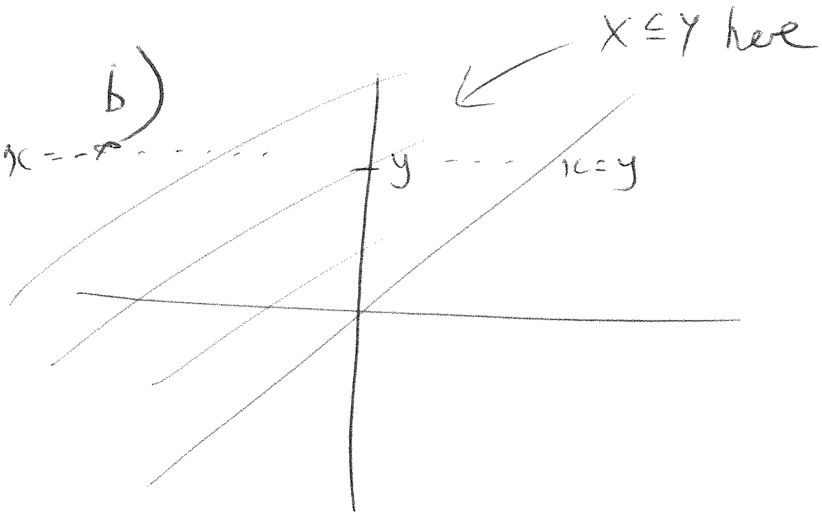
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{a-y} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) \left[\int_{-\infty}^{a-y} f_X(x)\,dx\right] dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y)\, F_X(a-y)\,dy$$

By definition of F_X, this is what's in square brackets in line above.

b)



$$P(X \leq Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{y} f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) \left[\int_{-\infty}^{y} f_X(x)\, dx \right] dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) F_X(y)\, dy.$$

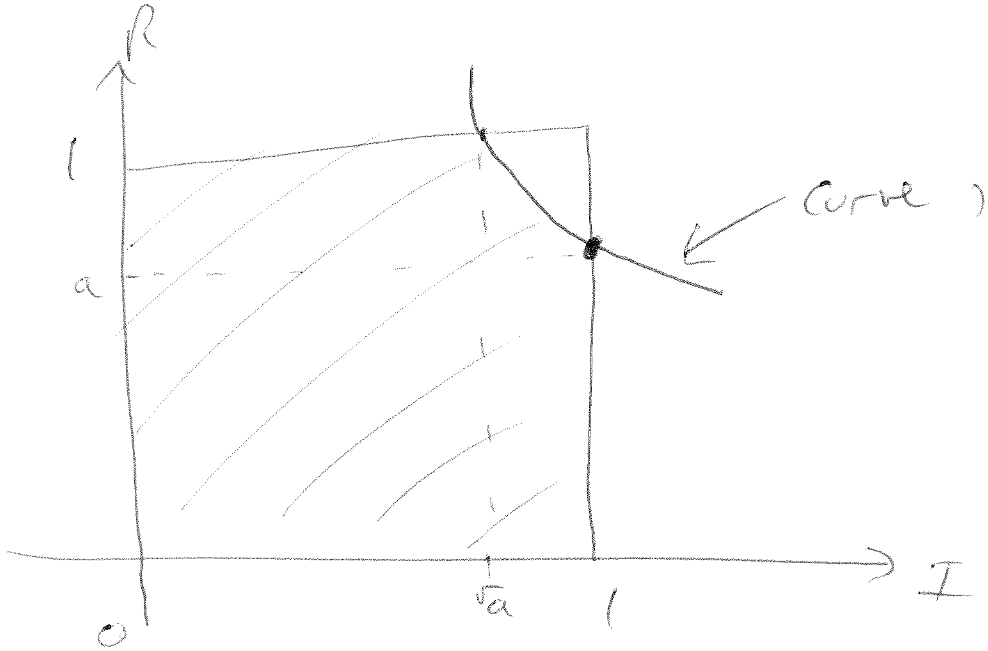
17) In office hours I talked about first finding density of I^2 (via density and distribution of I), but its easier to go at the problem directly.

Clearly, $f_W(x) = 0$ if $x < 0$ or $x > 1$.

For $0 \le a \le 1$, we first want

$F_W(a) = P(W \le a)$ (we'll then differentiate to get $f_W(a)$ in this range.)

To find $P(W = I^2 R \le a)$, we integrate the joint density of I and R over all points (in $I-R$ space) with $I^2 R \le a$



So we have to integrate the joint density over the shaded region above.

$$P(W \le a) = \int_0^{\sqrt{a}} \int_0^1 12xy(1-x)\,dy\,dx + \int_{\sqrt{a}}^1 \int_0^{a/x^2} 12xy(1-x)\,dy\,dx$$

By independence, joint density of I and R is product of densities.

I think that this integrates to

$$6a - a^2 - 2a\sqrt{a} - 2\sqrt{a}$$

(Not 100% sure; but it's 0 at 0, 1 at 1, which it should be)

The derivative of the wrt a is

$$6 - 2a - 3\sqrt{a} - \frac{1}{\sqrt{a}}$$

So $$f_W(x) = \begin{cases} 0 & \text{if } x < 0 \\ 6 - 2x - 3\sqrt{x} - \frac{1}{\sqrt{x}} & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$