MATH 30440, SPRING 2009 Homework 2 Solns

18) a) 5 ways to choose boy in 4^{th} position

14! ways to order everyone else

Probability : $\frac{5 \times 14!}{15!} = \frac{1}{3}$

b) Same as a)

c) 1 way to put particular boy in 3^{th} position

14! ways to order everyone else

Probability : $\frac{1 \times 14!}{15!} = \frac{1}{15}$

20) Experiment : list of length 4, each element of list is one of A, B, C, D (repetitions allowed)

(eg ABAB represents outcome : owners of first and third broken sets call repairman A, owners of second and fourth call B)

Total number of outcomes : $4^4 = 256$

Successful outcome : list which uses exactly two of the four symbols A, B, C, D.

There are $\binom{4}{2} = 6$ ways to decide on which two symbols (ie, which two repairmen) will be used (AB, AC, AD, BC, BD, CD)

For each such choice, there are 14 successful outcomes (eg, for AB there is:

$\left.\begin{array}{l} AAAB \\ AABA \\ ABAA \\ BAAA \end{array}\right\}$ A gets called out 3 times exactly

$\left.\begin{array}{l} BBBA \\ BBAB \\ BABB \\ ABBB \end{array}\right\}$ B gets called out 3 times exactly

$\left.\begin{array}{l} AABB \\ ABAB \\ ABBA \\ BAAB \\ BABA \\ BBAA \end{array}\right\}$ both get called out twice)

Total # successful outcomes: $\binom{4}{2} \times 14 = 84$

$P(\text{Successful outcome}) = \dfrac{84}{256}$

21) Discarding : $P(\text{Success on } k^{th} \text{ try})$

$$= \frac{(n-1)(n-2)(n-3)\cdots(n-(k-1)) \cdot 1}{n(n-1)(n-2)\cdots(n-(k-2))(n-(k-1))} = \frac{1}{n}$$

(this is for $k = 1, \ldots, n$; otherwise, probability is 0)

Denominator : # ways to pick out k keys from n, where no key is allow to be picked twice

Numerator : # ways to make the selection above, with the first, second, ..., $k-1^{th}$ choice all being the wrong key, and the k^{th} choice being the right key

Not discarding : Now trials are independent

$$P(\text{Success on } k^{th} \text{ trial}) = \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$\leftarrow$ Probability of success on k^{th} trial

Probability of failure on each of the first $k-1$ trials

Terms multiplied since trials are independent

(this is valid for $k = 1, 2, 3, \ldots$)

23) Experiment is: pick one of the three cards $\overset{①}{R}\overset{②}{R}$, $\overset{①}{R}\overset{②}{B}$, $\overset{①}{B}\overset{②}{B}$, and then pick one of the faces to display first; finally, display the other face.

$$S = \{ (\overset{①}{R}\overset{②}{R}, \overset{①}{R}, \overset{②}{R}), (\overset{①}{R}\overset{②}{R}, \overset{②}{R}, \overset{①}{R})$$
$$(\overset{①}{R}\overset{②}{B}, \overset{①}{R}, \overset{②}{B}), (\overset{①}{R}\overset{②}{B}, \overset{②}{B}, \overset{①}{R})$$
$$(\overset{①}{B}\overset{②}{B}, \overset{①}{B}, \overset{②}{B}), (\overset{①}{B}\overset{②}{B}, \overset{②}{B}, \overset{①}{B}) \}$$

$(\overset{①}{R}\overset{②}{B}, \overset{②}{B}, \overset{①}{R})$ means: Card which has R on one face (labeled ①) and B on other face (labeled ②) is chosen; face ② (B) is initially displayed; face ① (R) is the face that is hidden.

All six outcomes equally likely

Events: $A = \{$ Red face is displayed $\}$

$B = \{$ Red face is hidden $\}$

Want: $P(B|A) = \dfrac{P(BA)}{P(A)}$

Two of the six sample points are in BA (the first two listed) so $P(BA) = \frac{1}{3}$

Three of the six are in A (the first two and the third) so $P(A) = \frac{1}{2}$.

$$P(B|A) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

24) Four equally likely scenarios:

Eldest	Youngest
B	B
B	G
G	B
G	G

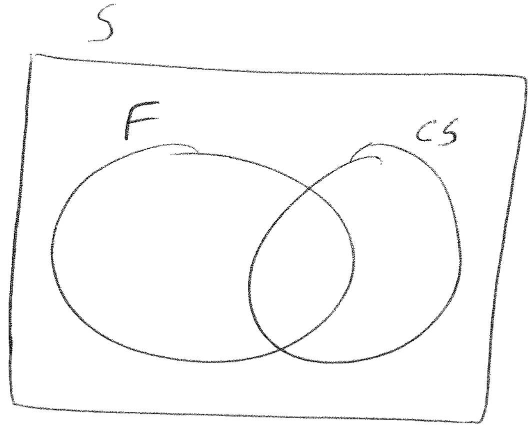
In two of these, a girl is eldest, so our reduced sample space has just ~~one~~ two (equally likely) points.

One of these two is GG, so probability $= \frac{1}{2}$

(Easier: assuming genders of successive children are independent,

$P(G \text{ second} | G \text{ first}) = P(G \text{ second}) = \frac{1}{2}$)

25)



$P(F) = .52$

$P(CS) = .05$

$P(FCS) = .02$

a) $P(F|CS) = \dfrac{P(FCS)}{P(CS)} = \dfrac{.02}{.05} = .4$

b) $P(CS|F) = \dfrac{P(FCS)}{P(F)} = \dfrac{.02}{.52} = \dfrac{1}{26}$

27) Experiment: Choose factory from A, B
test first radio, test second radio.

Possible outcomes:	Probabilities:
AWW	$\frac{1}{2} \times .95 \times .95 = .45125$
AWD	$\frac{1}{2} \times .95 \times .05 = .02375$
ADW	$\frac{1}{2} \times .05 \times .95 = .02375$
ADD	$\frac{1}{2} \times .05 \times .05 = .00125$
BWW	$\frac{1}{2} \times .99 \times .99 = .49005$
BWD	$\frac{1}{2} \times .99 \times .01 = .00495$
BDW	$\frac{1}{2} \times .01 \times .99 = .00495$
BDD	$\frac{1}{2} \times .01 \times .01 = .00005$

(ADW represents: radios came from A, first was defective, second was not)

The outcomes aren't equally likely. Eg,

$$P(AWD) = P(A)\,P(WD|A)$$
$$= \tfrac{1}{2} \times P(W|A)\,P(D|A) \quad \text{(independence)}$$
$$= \tfrac{1}{2} \times .95 \times .05 = .02375.$$

Events: $E = \{\text{First is defective}\}$

$F = \{\text{Second is defective}\}$

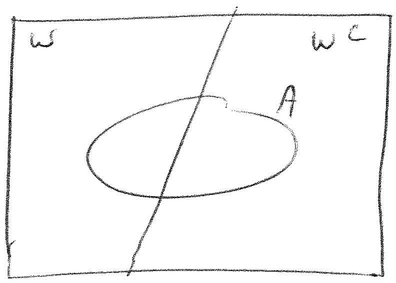
Want: $P(F|E) = \dfrac{P(FE)}{P(E)}$

$$= \frac{P(\{ADD, BDD\})}{P(\{ADW, ADD, BDW, BDD\})}$$

$$= \frac{.00125 + .00005}{.02375 + .00125 + .0495 + .00005}$$

$$= .04\dot{3}$$

29) $W = \{\text{Neighbour water}\}$
$A = \{\text{Plant alive}\}$



Given: $P(W) = .9$
$P(W^c) = .1$
$P(A|W) = .85$
$P(A|W^c) = .2$

a) $$\begin{aligned} P(A) &= P(AW) + P(AW^c) \\ &= P(W)P(A|W) + P(W^c)P(A|W^c) \\ &= .9 \times .85 + .1 \times .2 \\ &= .785 \end{aligned}$$

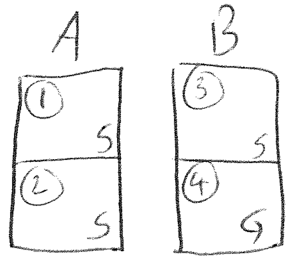
b) $$P(W^c|D) = \frac{P(W^cD)}{P(D)}$$

$P(D) = 1 - P(A) = .215$

$P(W^cD) = P(W^c)P(D|W^c) = .1 \times .8 = .08$

So $$P(W^c|D) = \frac{.08}{.215} = .372\ldots$$

33)

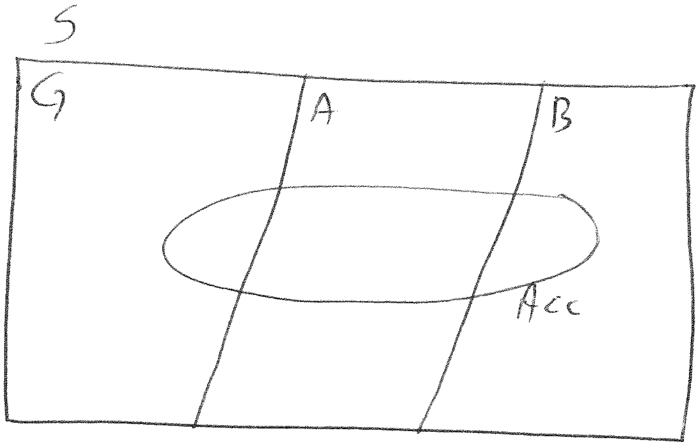


Four equally likely ~~events~~ outcomes:
Pick drawer 1, 2, 3, 4

$$P(\text{Silver in second drawer} \mid \text{Silver in first}) =$$

$$\frac{P(\text{Silver in both})}{P(\text{Silver in first})} = \frac{P(1 \text{ or } 2)}{P(1 \text{ or } 2 \text{ or } 3)} = \frac{2}{3}$$

35)



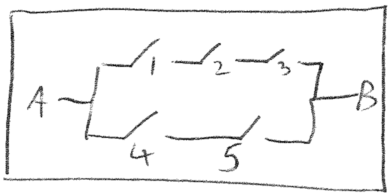
$P(G) = .2$
$P(A) = .5$
$P(B) = .3$
$P(Acc \mid G) = .05$
$P(Acc \mid A) = .15$
$P(Acc \mid B) = .3$

$$P(Acc) = P(Acc \mid G)P(G) + P(Acc \mid A)P(A) + P(Acc \mid B)P(B)$$
$$= .05 \times .2 + .15 \times .5 \times .3 \times .3$$
$$= .175 \quad (17.5\% \text{ have accidents})$$

$$P(G \mid \overset{No}{Acc}) = \frac{P(\overset{No}{Acc}\, G)}{P(\overset{No}{Acc})} = \frac{P(G)P(\overset{No}{Acc} \mid G)}{1 - P(Acc)} = .23$$

$$P(A \mid \overset{No}{Acc}) = \frac{P(A)P(\overset{No}{Acc} \mid A)}{P(\overset{No}{Acc})} = .515$$

37) a)



X through number indicates that relay is open

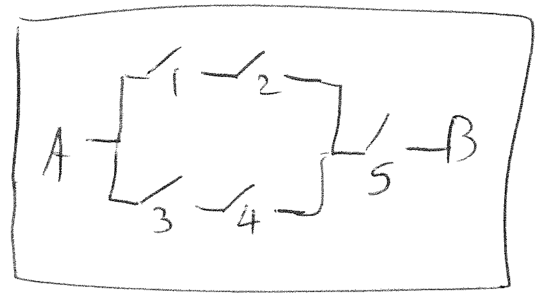
Good configurations	Probability
$1\ 2\ 3\ \cancel{4}\ \cancel{5}$	$p_1 p_2 p_3 (1-p_4)(1-p_5)$
$1\ 2\ 3\ 4\ \cancel{5}$	$p_1 p_2 p_3 p_4 (1-p_5)$
$1\ 2\ 3\ \cancel{4}\ 5$	$p_1 p_2 p_3 (1-p_4) p_5$
$1\ 2\ 3\ 4\ 5$	$p_1 p_2 p_3 p_4 p_5$
$\cancel{1}\ 2\ 3\ 4\ 5$	$(1-p_1) p_2 p_3 p_4 p_5$
$1\ \cancel{2}\ 3\ 4\ 5$	$p_1 (1-p_2) p_3 p_4 p_5$
$1\ 2\ \cancel{3}\ 4\ 5$	$p_1 p_2 (1-p_3) p_4 p_5$
$\cancel{1}\ \cancel{2}\ 3\ 4\ 5$	$(1-p_1)(1-p_2) p_3 p_4 p_5$
$\cancel{1}\ 2\ \cancel{3}\ 4\ 5$	$(1-p_1) p_2 (1-p_3) p_4 p_5$
$1\ \cancel{2}\ \cancel{3}\ 4\ 5$	$p_1 (1-p_2)(1-p_3) p_4 p_5$
$\cancel{1}\ \cancel{2}\ \cancel{3}\ 4\ 5$	$(1-p_1)(1-p_2)(1-p_3) p_4 p_5$

Probability is product by independence.

Sum of all these is probability of good circuit.

Easier: $\underbrace{p_1 p_2 p_3}_{\text{top closed}} + \underbrace{p_4 p_5}_{\text{bottom closed}} - \underbrace{p_1 p_2 p_3 p_4 p_5}_{\text{top and bottom closed}}$

b)



Good configurations :

$1\ 2\ \cancel{3}\ \cancel{4}\ 5 \rightarrow p_1 p_2 (1-p_3)(1-p_4) p_5$

$\downarrow$ etc.

$1\ 2\ 3\ \cancel{4}\ 5$

$1\ 2\ \cancel{3}\ 4\ 5$

$1\ 2\ 3\ 4\ 5$

$\cancel{1}\ 2\ 3\ 4\ 5$

$1\ \cancel{2}\ 3\ 4\ 5$

$\cancel{1}\ \cancel{2}\ 3\ 4\ 5$

Sum up these probabilities

Easier : $p_1 p_2 p_5 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5$

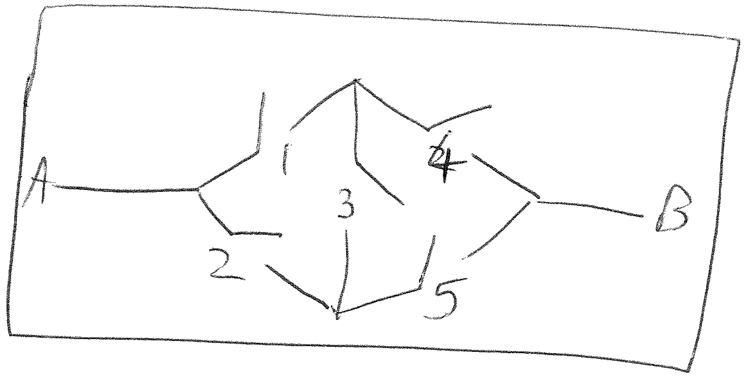
Prob. that 1, 2, 5 are closed

Prob. that 3, 4, 5 are closed

Probability that both possible paths are closed

(Using $P(E \cup F) = P(E) + P(F) - P(EF)$)

c)



Good configurations

with 3 open						with 3 closed				
1	~~2~~	~~3~~	4	~~5~~		1	2	3	4	5
1	2	~~3~~	4	~~5~~		1	2	3	4	~~5~~
1	~~2~~	~~3~~	4	5		1	2	3	~~4~~	5
1	2	~~3~~	4	5		1	~~2~~	3	4	5
1	2	~~3~~	~~4~~	5		~~1~~	2	3	4	5
~~1~~	2	~~3~~	4	5		~~1~~	2	3	~~4~~	5
~~1~~	2	~~3~~	~~4~~	5		~~1~~	2	3	4	~~5~~
						1	~~2~~	3	~~4~~	5
						1	~~2~~	3	4	~~5~~

Probability is sum of probabilities of these 16 configurations.

38) a) If W stands for working, F for not working, here are the configurations of the four components that lead to a working system:

1	2	3	4	
W	W	W	W	$\longrightarrow$ Probability $P_1P_2P_3P_4$
W	W	W	F	
W	W	F	W	
W	F	W	W	
F	W	W	W	
W	W	F	F	
W	F	W	F	$\longrightarrow$ Probability $P_1(1-P_2)P_3(1-P_4)$
F	W	W	F	
W	F	F	W	
F	W	F	W	
F	F	W	W	

Probability system is working is sum of these 11 probabilities.

b) Now there are 16 good configurations:

1	2	3	4	5
W	W	W	W	W
W	W	W	W	F
W	W	W	F	W
W	W	F	W	W
W	F	W	W	W
F	W	W	W	W
W	W	W	F	F
W	W	F	W	F
W	F	W	W	F
F	W	W	W	F
W	W	F	F	W
W	F	W	F	W
F	W	W	F	W
W	F	F	W	W
F	W	F	W	W
F	F	W	W	W

→ There are slicker ways to do Q37 and Q38, but:
It's better to be thorough and correct than slick and wrong. ←