7 Homework 5 (due Oct. 3) Name: SOLUTIONS

Note that the homework is due on **FRIDAY**! The purpose of this homework is to explore the how artificial variables expose redundancy in a system of equations, to practice using the full simplex method, and to explore the concept of duality.

Reading: Sections 3.7, 4.1, 4.2 and 4.3.

1. Problem set 3.7, question 2 (page 105).

   - part (a)

Constraints:

$$x_{11} + x_{12} + x_{13} = 350 \quad (1)$$
$$x_{21} + x_{22} + x_{23} = 550 \quad (2)$$
$$x_{11} + x_{21} = 275 \quad (3)$$
$$x_{12} + x_{22} = 325 \quad (4)$$
$$x_{13} + x_{23} = 300 \quad (5)$$

Linear relation revealing redundancy:

$$(1) + (2) = (3) + (4) + (5)$$

   - part (b) (For this part, if you use LP Assistant then print out your tableau and explain what part of the tableau exposes the redundancy).

Here's the tableau at the end of the artificial stage. Notice that the fourth constraint [with x_{11} basic] only mentions artificial variables, and so should be ignored in the real stage. This is where the redundancy is exposed.

x_2	1	1	0	0	0	-1	0	-1	1	1	0	50
x_6	-1	0	0	0	1	1	0	1	-1	0	0	275
x_3	0	0	1	0	0	1	1	1	-1	-1	0	300
x_4	1	0	0	1	0	0	0	0	1	0	0	275
x_{11}	0	0	0	0	0	0	-1	-1	1	1	1	0
	-7	0	0	0	0	3	-15	-9	-9	-7	0	-14950
	0	0	0	0	0	0	2	2	0	0	0	0

2. Problem set 3.7, question 3 (page 105). For both of these, if you use LP Assistant then print out your tableau. In the space below present the required answers.

- part (d)

In this part, after artificial stage is done, it is necessary to do a pivot to remove one artificial variable (at value 0) from the basis. There was no redundancy.

Optimum point: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 0$

Objective maximum : -6

Need to convert to a minimization to run simplex, but when recording final answer, it should be in terms of original maximization problem.

- part (g)

After artificial step, one basic variable is still in basis at value 0, but corresponding row has zero coefficients for all real variables; hence this problem exhibits redundancy.

Optimum point : $X_1 = 15$, $X_2 = 0 = X_3$, $X_4 = 10$,

Objective minimum : 25

3. Problem set 4.2, question 1 (page 130).

- part (a)

Minimize $100y_1 + 90y_2 + 500y_3$

subject to

$$5y_1 - y_2 \geq 20$$
$$-4y_1 + 12y_2 + y_3 \geq 30$$
$$y_1, y_2, y_3 \geq 0$$

- part (c)

First change second ine to $-3x_1 + 8x_2 \leq -10$

Dual: Minimize $60y_1 - 10y_2 + 20y_3$

subject to

$$5y_1 - 3y_2 + y_3 \geq -1$$
$$y_1 + 8y_2 + 7y_3 \geq 2$$

$y_1, y_2 \geq 0$, y_3 unrestricted [third primal constraint was =]

- part (f)

Many ways to do this → 1) Add constraints $y_3 \geq -15$ and $y_3 \leq 0$ to list of regular constraints, and declare y_3 "unrestricted"

2) Substitute $y_3' = -y_3$, add constraint $y_3' \leq 15$, declare $y_3' \geq 0$

3) Substitute $y_3'' = y_3 + 15$, add constraint $y_3'' \leq 15$, declare $y_3'' \geq 0$

Using 1), get primal:

Min $\boxed{2y_1 - 3y_2 + 4y_3}$

st

$$8y_1 - y_3 = 50$$
$$-6y_2 - y_3 \geq -60$$
$$y_3 \geq -15$$
$$-y_3 \geq 0$$

$y_1, y_2 \geq 0$, y_3 unrestricted

and so dual: Max $50x_1 - 60x_2 - 15x_3$

subject to

$$8x_1 \leq 2$$
$$-6x_2 \leq -3$$
$$-x_1 - x_2 + x_3 - x_4 \leq 4$$

x_1 unrestricted, $x_2, x_3, x_4 \geq 0$

4. Problem set 4.2, question 2, part (d) (page 131).

Primal is equivalent to :

$$\text{Max} \quad x_1 - 2x_2 - 3x_3 \quad \text{s.t.} \qquad \begin{aligned} -x_2 - 2x_3 &\le -1 \\ x_1 \qquad + 3x_3 &\le 2 \\ 2x_1 - 3x_2 \qquad &= 3 \end{aligned}$$

$x_1, x_2 \ge 0$, x_3 unrestricted

The dual of this problem is

$$\text{Min} \quad -y_1 + 2y_2 + 3y_3 \quad \text{s.t} \qquad \begin{aligned} y_2 + 2y_3 &\ge 1 \\ -y_1 \qquad - 3y_3 &\ge -2 \\ -2y_1 + 3y_2 \qquad &\ge -3 \end{aligned}$$

$y_1, y_2 \ge 0$, y_3 unrestricted,

which is equivalent to

$$\text{Max} \quad y_1 - 2y_2 - 3y_3 \quad \text{subject to}$$

$$\begin{aligned} y_2 + 2y_3 &\ge 1 \\ y_1 \qquad + 3y_3 &\le 2 \\ 2y_1 - 3y_2 \qquad &= 3 \end{aligned}$$

$y_1, y_2 \ge 0$, y_3 unrestricted.

This is <u>exactly</u> the primal problem, as it was required to show.

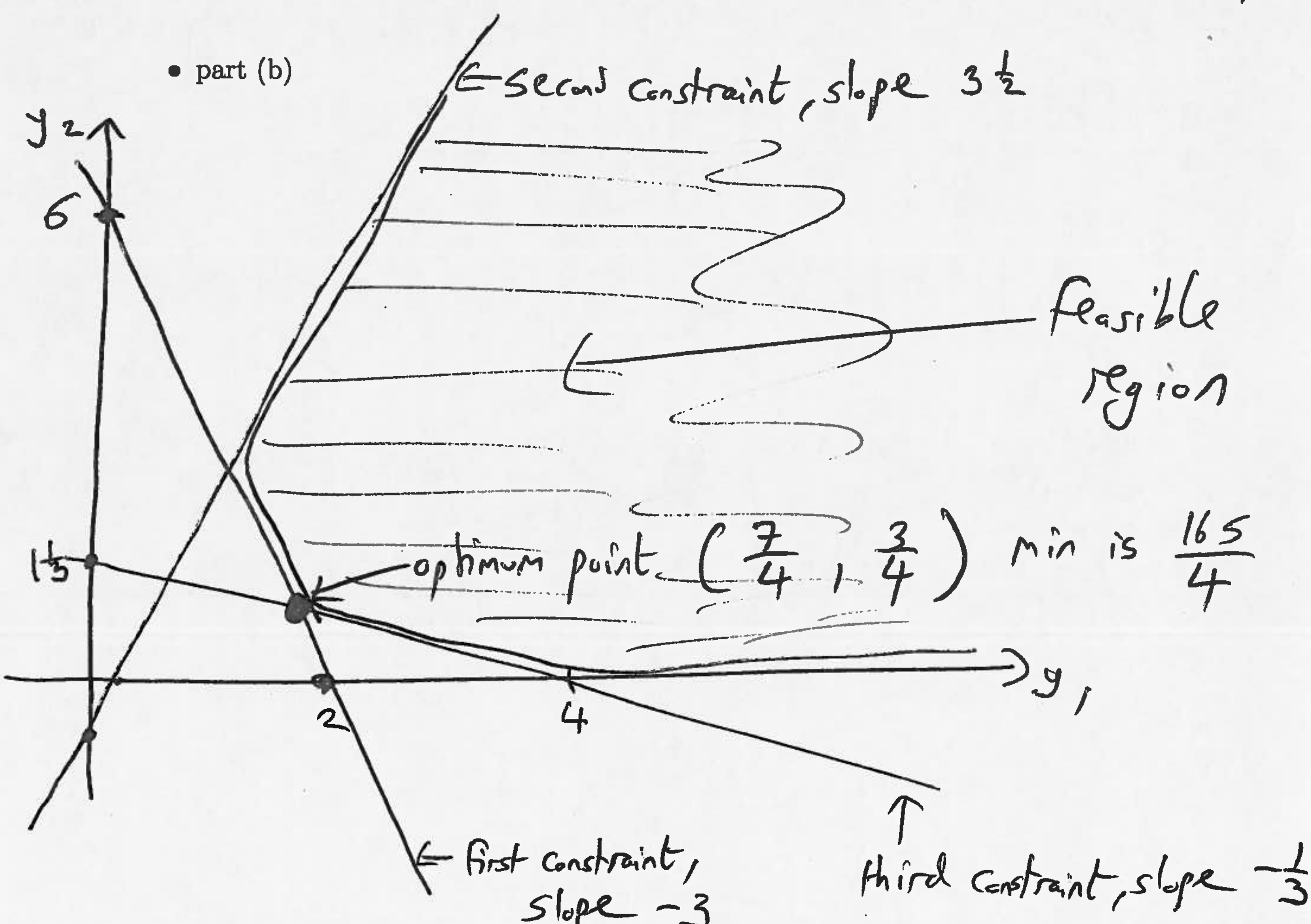
5. Problem set 4.2, question 3 (page 131)

- part (a)

Dual problem is to minimize $15y_1 + 20y_2$,
with $y_1, y_2 \geq 0$;
So dual objective will always be $\underline{\underline{\geq 0}}$

(and so is bounded from below)

- part (b)



Slope of objective line is $-\frac{3}{4}$, between that of first + third constraints, so min is at intersection of these two lines,

- part (c) (If you use LP Assistant then print out your tableau. In the space below present the required answers.)

Optimum for maximization problem is $\frac{165}{4}$

[Same as for dual, minimization, problem]

Optimum point: $x_1 = \frac{25}{8}$, $x_2 = 0$, $x_3 = \frac{45}{8}$

- part (d)

Slack variable 1: objective row entry is $\frac{7}{4}$

" " 2: " " " " $\frac{3}{4}$;

Compare: $\left(\frac{7}{4}, \frac{3}{4}\right)$ was optimum point for dual problem!

6. Problem set 4.3, question 3 (page 136)

- part (a)

This is directly from definition of dual.

- part (b)

Manufacturer can <u>either</u> use 50 lbs of aluminium, 6 minutes of machine time, and 3 hours of labour to produce a rowboat and earn \$ 50 revenue, or sell those resources to the competitor and net

$$50y_1 + 6y_2 + 3y_3 \text{ dollars revenue;}$$

for it to be worth the manufacturer's consideration, must have $50y_1 + 6y_2 + 3y_3 \geq 50$.

Same consideration for canoes leads to constraint

$$30y_1 + 5y_2 + 5y_3 \geq 60.$$

Also, $y_1, y_2, y_3 \geq 0$.

Competitor's objective: Minimize his total outlay,

$$\underbrace{2000y_1}_{\text{purchase of aluminium}} + \underbrace{300y_2}_{\text{machine time}} + \underbrace{200y_3}_{\text{labor.}}$$

Conclusion: Competitor's problem = dual of manufacturer's.

- part (c) (If you use LP Assistant then print out your tableau. In the space below present the required answers.)

Solved via simplex after adding (negative signed) slack variables, and using method of artificial variables.

Optimum outlay : \$ 2750

Achieved at : $y_1 = \frac{7}{16}$

$y_2 = 0$

$y_3 = \frac{75}{8}$

see p. 300 of textbook

Manufacturer's max profit is exactly \$ 2750, so competitor can set prices to give a favorable (but not better) offer

- part (d)

Manufacturer's max profit is achieved by:

R = # rowboats produced = 25

C = # canoes " = 25

(again, see p. 300 of text)

Objective row coefficient of slack variable 1 is $\underline{-25}$

" " " " " " 2 is $\underline{-25}$

→ Numbers are the same, up to sign.

7. Problem set 4.3, question 5 (page 138)

- part (a) R = # radios produced, T = # TV's, S = # stereos

Maximize $8R + 60T + 45S$ s.t

$$2R + 12T + 15S \le 1500$$

$$R + 8T + 6S \le 920$$

$$R, S, T \ge 0$$

- part (b)

Dual: Minimize $1500y_1 + 920y_2$

subject to:

$$2y_1 + y_2 \ge 8$$

$$12y_1 + 8y_2 \ge 60$$

$$15y_1 + 6y_2 \ge 45$$

$$y_1, y_2 \ge 0$$

- part (c)

Problem of determining the value of unit of labor, unit of material, ~~is~~ involves exactly the 3 constraints of the dual of the profit maximization problem. Total value of available materials, labor is objective of dual problem. Minimization in dual relates to fact that manager does not want to overestimate value.