2 Homework 1 (due Sept. 3) Name: SOLUTIONS

The purpose of this homework is to illustrate the variety of situations that can be modelled with a linear objective functions and linear constraint. [It's implicit in all the questions that "formulate mathematically" means "formulate as the problem of maximizing/minimizing a **linear** objective function, subject to a collection of **linear** constraints on the variables"]

Reading: Sections 1.1 1.2 and 1.4 (first half of page), and 2.1, 2.2, 2.3, 2.4 and 2.6.

1. Problem set 2.2, question 7 (page 18)

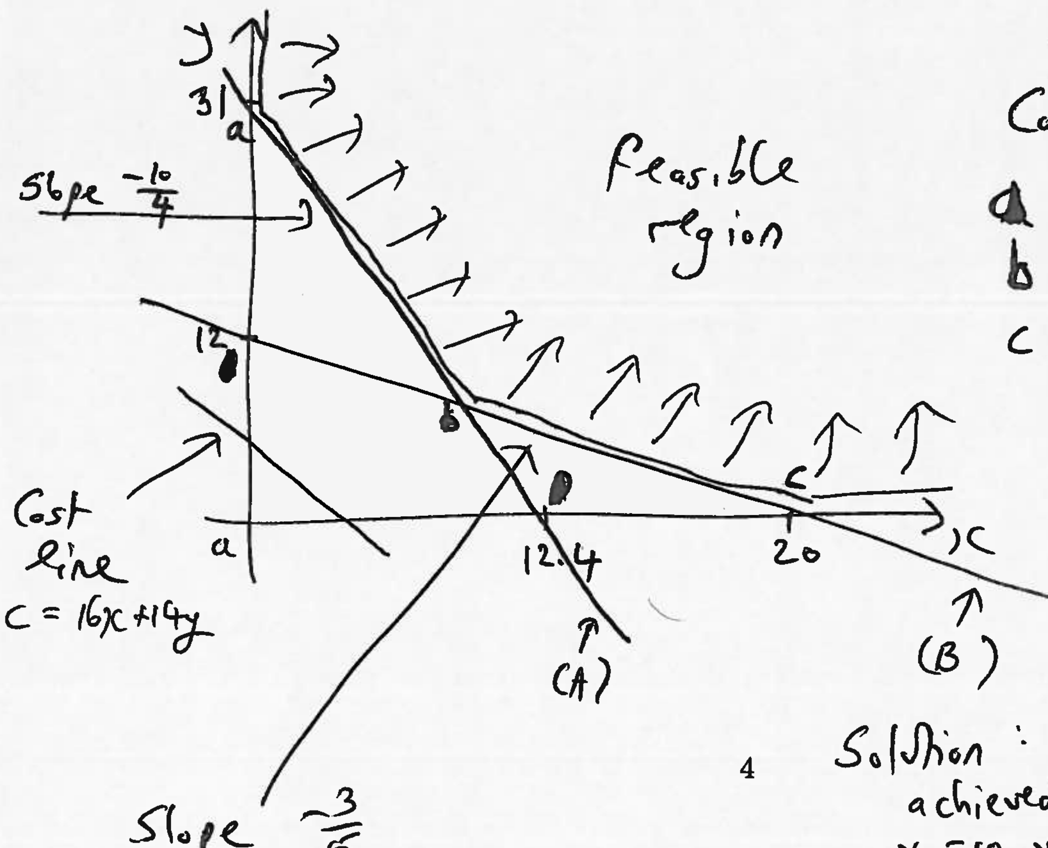
   - part (a) [note that you are required both to set up and solve the problem]

x = # lbs of feed 1, y = # lbs of feed 2

Formulation : Minimize cost $c = 16x + 14y$ subject to

$$10x + 4y \geq 124 \quad (A)$$
$$3x + 5y \geq 60 \quad (B)$$
$$x, y \geq 0.$$

Graphical solution :



Corner point analysis :

a (0, 31)	cost	434
b (10, 6)	cost	244
c (20, 0)	cost	320

Solution : Minimum cost is achieved at corner point b, $x = 10, y = 6$, cost = 244

- part (b)

If cost of feed 1 is c_1, and cost of feed 2 is c_2, then cost line is $C = c_1 x + c_2 y$, slope $\frac{-c_1}{c_2}$

Corner point b $(10, 6)$ is optimal as long as

$$\frac{-10}{4} \le \frac{-c_1}{c_2} \le \frac{-3}{5}, \quad \text{or}$$

$$\frac{3}{5} \le \frac{c_1}{c_2} \le \frac{5}{2}$$

[Reality check : If $c_1 = 16$, $c_2 = 14$, then $\frac{c_1}{c_2}$ is indeed in this range]

- part (c)

There will be a non-unique solution exactly when the cost line is parallel to one of the bounding lines of the feasible region.

Case i) $\frac{-c_1}{c_2} = -\infty$ [Cost line $\parallel$ to line segment joining a to $(0, \infty)$; this is the degenerate case where $c_2 = 0$]

so $\frac{c_1}{c_2} = \infty$

Case ii) $\frac{c_1}{c_2} = \frac{10}{4} = \frac{5}{2}$ [Cost line $\parallel$ to (A)]

Case iii) $\frac{c_1}{c_2} = \frac{3}{5}$ [Cost line $\parallel$ to (B)]

Case iv) $\frac{c_1}{c_2} = 0$ [Cost line $\parallel$ to x-axis; degenerate case where $c_1 = 0$, ie feed 1 is free]

2. Problem set 2.2, question 14 (page 20) [Be careful! If you find that using 0 of each product is feasible, you haven't set the problem up correctly! Also, note that this question only requires you to set up the problem]

Let x = amount of A used per gallon

y = " " B " "

z = " " C " "

Formulation:

Minimize cost $C = 1.6x + .5y + 1.4z$

[notice this is cost/gallon]

Subject to

$60x + 18y + 75z \geq 50$ (antifreeze)

$10x + 3y \geq 5$ (additives)

$x + y + z = 1$ [all units are per gallon; this constraint says that we make exactly one gallon]

$x, y, z \geq 0$

[By thinking about cost per gallon, we avoid the incorrect solution of mixing 0 parts A to 0 parts B to 0 parts C to get a mixture costing 0]

Note: Since graphical solution to problem without considering integral constraints gives integral optimal solution, this is also optimal solution to problem with integral constraints.

3. Problem set 2.3, question 10 (page 28) [note that you are required both to set up and solve the problem]

Two decision variables: x = total # of cabinets installed
y = # of cabinets installed using a purchased frame.

Income: $\underbrace{100x}_{\text{sales}} - \underbrace{(6\times5)x}_{\text{labor}} - \underbrace{(5\times3)x}_{\text{wood}} - \underbrace{27y}_{\text{frames}} - \underbrace{(6\times2)(x-y)}_{\text{labor for locally constructed frames}} - \underbrace{(5\times1)(x-y)}_{\text{wood for locally constructed frames}}$

$= 38x - 10y$

Formulation: Maximize profit $p = 38x - 10y$ subject to

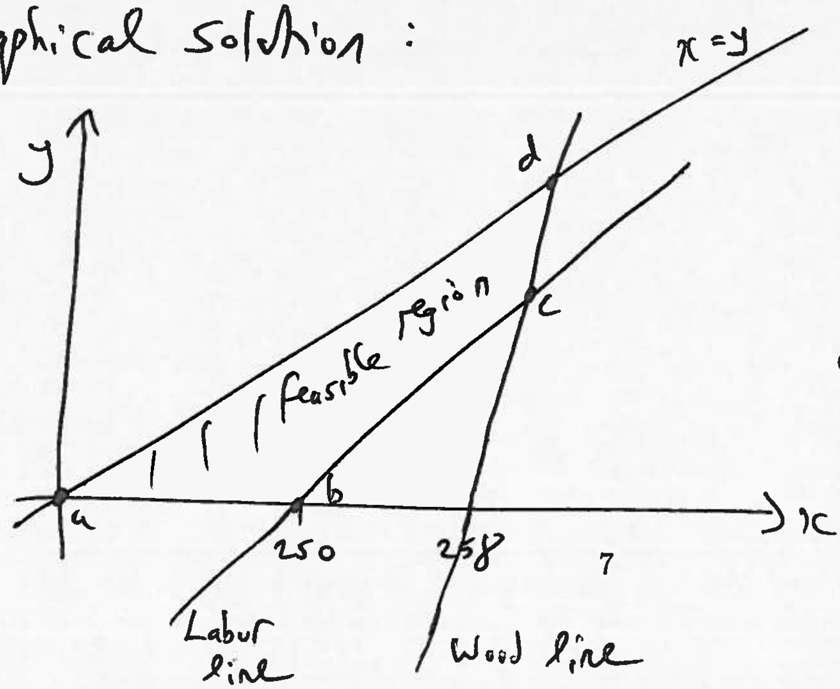
Labor constraint: $5x + 2(x-y) \le 1750$, OR $7x - 2y \le 1750$

Wood constraint: $\underbrace{3x}_{\text{installation}} + \underbrace{1(x-y)}_{\text{frames}} \le 1032$, OR $4x - y \le 1032$

Natural purchased frames constraint: $y \le x$ [Not actually necessary]

$x, y \ge 0$, $x, y \in \mathbb{Z}$

Graphical solution:



Corner points:

a $(0,0)$ $p = 0$

b $(250,0)$ $p = 9500$

c $(314, 224)$ $p = 9692$

d $(344, 344)$ $p = 9632$

Solution: Make 314 cabinets of which 224 have bought-in frames, for max profit of \$9692.

4. Problem set 2.3, question 15 (page 30) [note that you are required only to set up the problem]

Decesion variables:

a = # of arrangements of type A made
b = # " " B "
c = # " " C "
x = # local carnations purchased
y = # local roses purchaced
z = # distant carnations purchased

I'm choosing to use 1 flower as unit, not 1 dozen!

Formulation: Maximize income I =

$$\underbrace{2.75a + 6.5b + 5.25c}_{\text{revenue}} \underbrace{- \frac{1.8x}{12} - \frac{4.8y}{12} - \frac{3}{12}z}_{\text{Materials cost}}$$

$$= 2.6x + 6.1y + 5z$$ subject to

Production constraints: $5a + 12b + 3c \leq x + z$ (Carnations)
$2a + 4b + 6c \leq y$ (Roses)

Availibility constraints: $x \leq 12 \times 85$
$y \leq 12 \times 75$
$z \leq 12 \times 65$,

and $a, b, c, x, y, z \geq 0$ and all $\in \mathbb{Z}$.

5. Problem set 2.4, question 4 (page 37) [note that you are required only to set up the problem]

Decision variables : x_{ij} = # units shipped from warehouse i to outlet j, $i \in \{1, 2, 3\}$, $j \in \{1, 2, 3, 4$

Objective is to minimize cost $C =$

$$\left.\begin{aligned} &12x_{11} + 15x_{12} + 10x_{13} + 25x_{14} \\ +\ &10x_{21} + 19x_{22} + 11x_{23} + 30x_{24} \\ +\ &21x_{31} + 30x_{32} + 18x_{33} + 40x_{34} \end{aligned}\right\} \text{Shipping costs}$$

$$\left.\begin{aligned} &+ 6[100 - x_{11} - x_{12} - x_{13} - x_{14}] \\ &+ 6[150 - x_{21} - x_{22} - x_{23} - x_{24}] \\ &+ 12[300 - x_{31} - x_{32} - x_{33} - x_{34}] \end{aligned}\right\} \text{Storage}$$

Subject to :

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &\le 100 \\ x_{21} + x_{22} + x_{23} + x_{24} &\le 150 \\ x_{31} + x_{32} + x_{33} + x_{34} &\le 300 \end{aligned}$$

each warehouse can't give more than it has

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 120 \\ x_{12} + x_{22} + x_{32} &= 120 \\ x_{13} + x_{23} + x_{33} &= 120 \\ x_{14} + x_{24} + x_{34} &= 120 \end{aligned}$$

each outlet receives <u>exactly</u> 120 units

as well as : $x_{12} \le x_{22}$ (Contract constraint 1)

$x_{34} \ge 60$ (" " 2)

plus : all $x_{ij} \ge 0$, all integral.

6. Problem set 2.4, question 7 (page 38) [note that you are required only to set up the problem]

Decision variables : x_{ij} = # units shipped from center i to store j, $i \in \{1,2,3\}$, $j \in \{1,2,3,4\}$

S_2 = surplus at center 2 [not really a new variable, since it can be calculated from the x_{ij}'s, but it might be cleaner to include it]

S_3^a = surplus at center 3 sold at \$27/unit
S_3^b = " " " " " " \$23/unit

Formulation : for convenience, let C_{ij} = cost per unit to move from i to j (so eg $C_{13} = 65$)

$$\text{Minimize Cost } C = \underbrace{\sum_{(i,j)} C_{ij} x_{ij}}_{\text{shipping}} \underbrace{- 25S_2 - 27S_3^a - 23S_3^b}_{\text{minus revenue}}$$

subject to :

$$x_{11} + x_{12} + x_{13} + x_{14} = 225$$
$$x_{21} + x_{22} + x_{23} + x_{24} + S_2 = 300$$
$$x_{31} + x_{32} + x_{33} + x_{34} + S_3^a + S_3^b = 375$$
$$x_{1j} + x_{2j} + x_{3j} = 200 \quad \text{for } j = 1,2,3,4$$
$$S_3^a \leq 30,$$

all variables integer, ≥ 0.

Note that this is all that is needed to ensure that in any optimal solution, $S_3^b > 0$ only if $S_3^a = 30$; no optimal solution will sell surplus cheaply when there is an option to sell for higher price.

7. **Extra credit problem!** Set up (just set up) the two problems concerning queens on a chessboard (from the handout on August 27, also on the website) as problems of optimizing some linear function of some variables, subject to some linear constraints (and perhaps some constraints to say that some variables are integers). So that things don't get ridiculously out-of-hand, just do the problems for a 3-by-3 chessboard.

First problem: one variable for each square of board, telling how many queens on that square (variable names a–i, as shown)

a	b	c
d	e	f
g	h	i

objective: minimize $a+b+c+d+e+f+g+h+i$ (# queens)

subject to 1) one constraint for each square, saying square is either occupied or attacked

eg for square labeled f:

$$f+c+i+e+d+b+h \geq 1$$

2) three constraint for each square, saying there is either 0 or 1 queen on that square

eg for square labeled f:

$$f \geq 0 \qquad f \in \mathbb{Z}$$
$$f \leq 1$$

Second problem: in addition to above, add one constraint for each row, column and diagonal, saying that there are not two queens on that row, column or diagonal

eg for row a,b,c: $a+b+c \leq 1$

for diagonal f,h: $f+h \leq 1$