Economic interpretation of dual

Consider the following primal problem:

Maximize

$$c_1x_1 + \ldots + c_nx_n$$

subject to all $x_i \geq 0$

$$\begin{aligned} a_{11}x_1 + \ldots a_{1n}x_n &\leq b_1 \\ &\ldots \\ a_{m1}x_1 + \ldots a_{mn}x_n &\leq b_m. \end{aligned}$$

Economic interpretation:

n economic *activities*, m *resources*

c_j is *revenue per unit* of activity j

b_i is *maximum availability* of resource i

a_{ij} is *consumption of resource* i *per unit of activity* j

The dual

Minimize

$$b_1 y_1 + \ldots + b_m y_m$$

subject to all $y_i \geq 0$

$$\begin{aligned} a_{11} y_1 + \ldots a_{1m} y_m &\geq c_1 \\ &\ldots \\ a_{n1} x_1 + \ldots a_{nm} y_m &\geq c_n. \end{aligned}$$

Interpreting the dual variables

If $(x_1, \ldots, x_n)$ is optimal for the primal, and $(y_1, \ldots, y_m)$ is optimal for the dual, then we know:

$$c_1 x_1 + \ldots + c_n x_n = b_1 y_1 + \ldots + b_m y_m$$

Left-hand side: Maximal revenue

Right-hand side:

$\sum_{\text{resources } i}$ (availibility of resource i) $\times$ (revenue per unit of resource i)

In other words: Value of y_i at optimal is *dual price of resource* i

Away from optimality, we have

$$c_1 x_1 + \ldots + c_n x_n < b_1 y_1 + \ldots + b_m y_m$$

Left-hand side: current (suboptimal) revenue

Right-hand side: $\sum_{\text{resources } i}$ (worth of resource i)

Solution is not optimal because resources are not being fully utilized

Interpreting the dual constraints

If $(x_1, \ldots, x_n)$ is feasible (not necessarily optimal) for the primal, and $(y_1, \ldots, y_m)$ is the corresponding collection of dual values, then we know:

Current objective coefficient of x_j

$$= \text{(Left-hand side of dual constraint } j) - \text{(Right-hand side)}$$
$$= (a_{1j}y_1 + \ldots + a_{mj}y_j) - c_j$$

c_j is a measure of revenue per unit (of activity j)
So $a_{1j}y_1 + \ldots + a_{mj}y_j$ is an *imputed* (implicit) *cost per unit* (of act. j)

Also, y_i is *imputed cost* per unit of resource i in a unit of activity j

If objective coefficient of x_j (= cost − revenue, = *reduced cost*) is **strictly negative**, then revenue > cost, so it makes since to increase activity j — this is the pivoting process of the simplex method