Relations between Primal and Dual

If the primal problem is

Maximize $c^t x$ subject to $Ax = b$, $x \geq 0$

then the dual is

Minimize $b^t y$ subject to $A^t y \geq c$ (and y unrestricted)

Easy fact:

If x is feasible for the primal, and y is feasible for the dual, then

$$c^t x \leq b^t y$$

So (primal optimal) $\leq$ (dual optimal) (**Weak Duality Theorem**)

Much less easy fact: (**Strong Duality Theorem**)

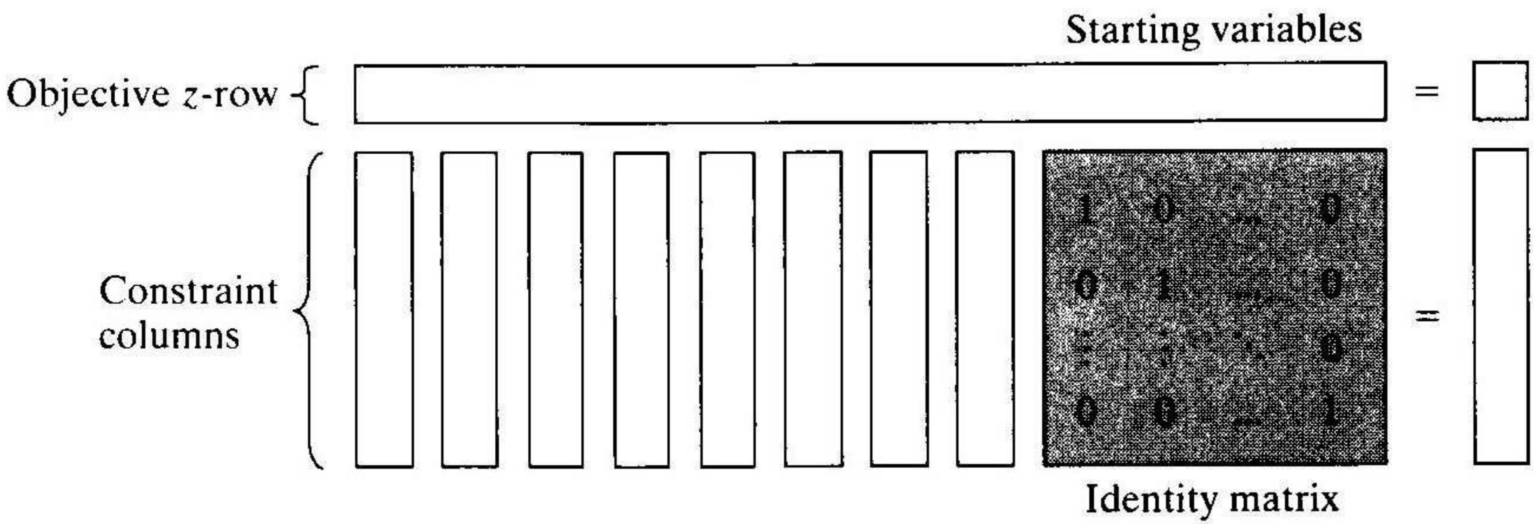
If one of the primal and the dual have finite optima, they both have and

$$\text{(primal optimal)} = \text{(dual optimal)}$$

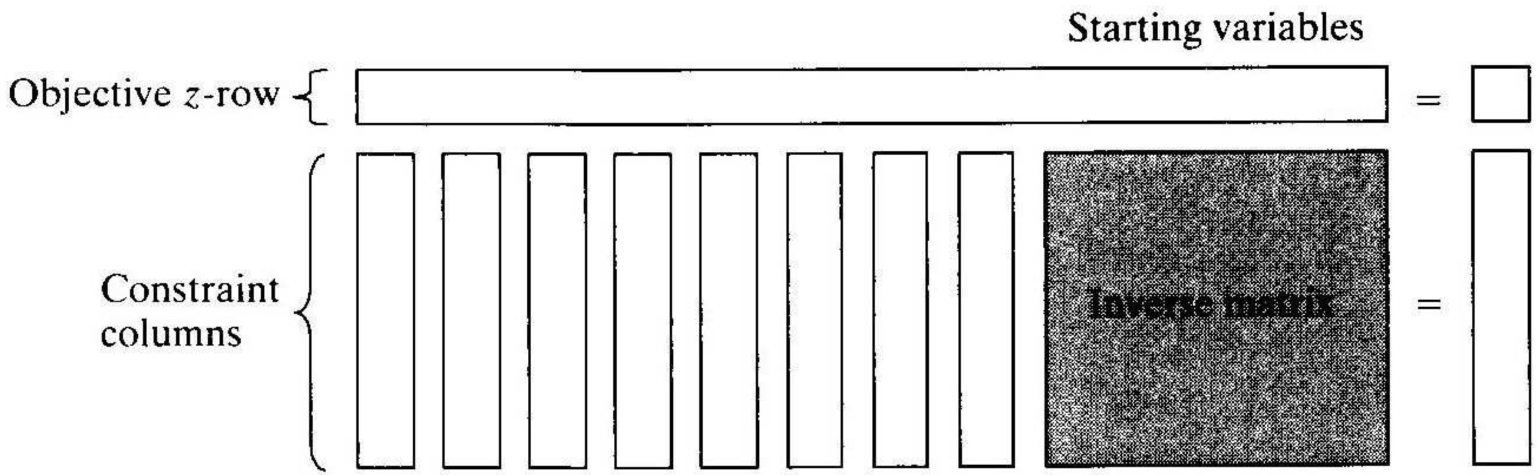
The “Inverse Matrix”

In the initial simplex tableau, there’s an identity matrix. At a later simplex tableau, the “inverse matrix” is the matrix occupying the same space as that original identity matrix.

The inverse matrix conveys *all* information about the current state of the algorithm, as we will see.



(Starting tableau)



(General iteration)

Computing dual values from Inverse Matrix

If we have reached the optimal primal tableau, these methods give the optimal dual values; at earlier iterations, they give a certain “dual” of the current basic feasible solution

Method 1:
Row vector of dual values = Row vector of *original objective values* of current basic variables (listed in order they appear along basic column of current tableau) X current inverse

Method 2: (see textbook for this)
Write w_i for *initial basic variable* in row i.
Value of dual variable y_i = current z-row coefficient of w_i + *original objective coefficient* of w_i

Example — Primal problem

Minimize $4x_1 + 2x_2 - x_3$

subject to

$x_1 + x_2 + 2x_3 \geq 3$

$2x_1 - 2x_2 + 4x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

In standard form:

Minimize $4x_1 + 2x_2 - x_3 + 0s_1 + MR + 0s_2$

subject to

$x_1 + x_2 + 2x_3 - s_1 + R = 3$

$2x_1 - 2x_2 + 4x_3 + s_2 = 5$

$x_1, x_2, x_3, s_1, R, s_2 \geq 0.$

Example — The dual

Maximize $3y_1 + 5y_2$

subject to

$y_1 + 2y_2 \leq 4$

$y_1 - 2y_2 \leq 2$

$2y_1 + 4y_2 \leq -1$

$-y_1 \leq 0$, i.e. $y_1 \geq 0$

$y_1 \leq M$

$y_2 \leq 0$

Note that the addition of the artificial variable to the primal adds a new constraint to the dual: $y_1 \leq M$. But since we imagine M to be very large, this effectively puts no new constraint on y_1. For convenience, we'll take $M = 100$.

Example — initial tableau

	x_1	x_2	x_3	s_1	R	s_2	soln
z-row	96	98	201	-100	0	0	300
R	1	1	2	-1	**1**	**0**	3
s_2	2	-2	4	0	**0**	**1**	5

Inverse matrix is bolded

Current solution: $R = 3$, $s_2 = 5$, $z = 300$. Feasible, but not optimal

Current "dual solution":

$$(y_1\ y_2) = (100\ 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (100\ 0)$$

i.e. $y_1 = 100, y_2 = 0, z_{dual} = 300$
We will see later that this is "optimal but not feasible".

Example — first iteration

	x_1	x_2	x_3	s_1	R	s_2	soln
z-row	-4.5	198.5	0	-100	0	-50.25	48.75
R	0	2	0	-1	$\mathbf{1}$	$\mathbf{-.5}$	$.5$
x_3	$.5$	$-.5$	1	0	$\mathbf{0}$	$\mathbf{.25}$	1.25

Inverse matrix is bolded

Current solution: $R = .5, x_3 = .25$, $z = 48.75$. Feasible, but not optimal

Current "dual solution":

$$(y_1 \; y_2) = (100 \; -1) \begin{pmatrix} 1 & -.5 \\ 0 & .25 \end{pmatrix} = (100 \; -50.25)$$

i.e. $y_1 = 100, y_2 = -50.25, z_{dual} = 48.75$
Again, "optimal but not feasible".

Example — optimal tableau

	x_1	x_2	x_3	s_1	R	s_2	soln
z-row	-4.5	0	0	$-.75$	-99.25	$-.625$	$-.875$
x_2	0	1	0	$-.5$	$\mathbf{.5}$	$\mathbf{-.25}$	$.25$
x_3	$.5$	0	1	$-.25$	$\mathbf{.25}$	$\mathbf{.125}$	1.375

Inverse matrix is bolded

Optimal solution: $x_2 = .25, x_3 = 1.375$, $z = -.875$.

Optimal dual solution:

$$(y_1\ y_2) = (2\ \ -1)\begin{pmatrix} .5 & -.25 \\ .25 & .125 \end{pmatrix} = (.75\ \ -.625)$$

i.e. $y_1 = .75, y_2 = -.625, z_{dual} = -.875$
This is readily checked to be feasible and optimal for dual

Getting whole tableau from Inverse (and initial data)

Constraint columns:

New constraint column = current inverse X original constraint column

Objective coefficients:

Objective coefficient (z-row entry) of variable x_j = Left hand side of jth dual constraint (evaluated at current "dual solution") - right hand side of jth dual constraint

Example — first iteration of previous problem

	x_1	x_2	x_3	s_1	R	s_2	soln
z-row	-4.5	$\mathbf{198.5}$	0	-100	0	-50.25	48.75
R	0	$\mathbf{2}$	0	-1	$\mathbf{1}$	$\mathbf{-.5}$	$.5$
x_3	$.5$	$\mathbf{-.5}$	1	0	$\mathbf{0}$	$\mathbf{.25}$	1.25

We look at (bolded) x_2 constraint column and objective entry

$$\begin{pmatrix} 1 & -.5 \\ 0 & .25 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -.5 \end{pmatrix}$$

(Inverse X original x_2 column = new x_2 column)
Coefficient of x_2 in z-row is computed by

$$(y_1 - 2y_2) - 2 = ([100] - 2[-50.25]) - 2 = 198.5$$

using values of y_1, y_2 computed earlier.