Converting an LP to standard form

All LP solvers first convert the given program to *standard form* which means

- all variables involved are restricted to be non-negative
- all constraints are equalities, with constant, non-negative right-hand sides

Converting may require new variables and rearranging constraints:

- an inequality can be multiplied by -1 to get non-negative rhs
- inequalities can be converted to equalities by adding or subtracting non-negative slack variables
- Unrestricted variables can be dealt with by writing the variable as the difference of two new non-negative variables

Example 1: the meatloaf problem

Recall the meatloaf problem, whose formulation was

Minimize

$$80x + 60y$$

subject to

$$\begin{aligned} x + y &\geq 1 \\ -.05x + .07y &\leq 0 \\ x,\ y &\geq 0. \end{aligned}$$

To convert to standard form, we introduce two new variables, $s_1 \geq 0$ and $s_2 \geq 0$. The first measures how much over 1 the quantity $x + y$ is, and the second measures how much under 0 the quantity $-.05x + .07y$ is.

The meatloaf problem in standard form

Minimize

$$80x + 60y$$

subject to

$$\begin{aligned} x + y - s_1 &= 1 \\ -.05x + .07y + s_2 &= 0 \\ x,\ y,\ s_1,\ s_2 &\geq 0. \end{aligned}$$

Note that if (x, y, s_1, s_2) is feasible for this problem, then (x, y) is feasible for the original; and if (x, y) is feasible for the original, then $(x, y, (x + y) - 1, 0 - (-.05x + .07y))$ is feasible for this problem. Since the objective only involves x and y, the two problems have the same solution.

Example 2: production without overtime

A company manufactures two products, A and B. The relevant production data is as follows

- Profit per unit: $2 and $5 respectively
- Labor time per unit: 2 hours and 1 hour respectively
- Machine time per unit: 1 hour and 2 hours respectively
- Available labor and machine time: 80 hours and 65 hours respectively

An easy linear program to maximize profit is

Maximize $2x_A + 5x_B$

subject to $2x_A + x_B + s_1 = 80$

$$x_A + 2x_B + s_2 = 65$$

where $x_A, x_B \geq 0$ are amounts of A and B produced, respectively, and $s_1, s_2 \geq 0$.

Example 3: production with overtime

Consider the same problem as before, but now with the wrinkle that labor and machine overtime may be purchased at a cost:

- Labor and machine overtime cost: \$15 and \$10 per hour, respectively

Now the labor constraint is

$$2x_A + x_B + s_1 = 80$$

with s_1 unrestricted. It may possibly be positive (representing unused labor) or negative (representing overtime used). We have a similar unrestricted variable s_2 for the machine constraint, and the objective becomes the unwieldy (and non-linear)

$2x_A + 5x_B$ (if s_1, s_2 both positive)

$2x_A + 5x_B + 15s_1$ (if s_1 negative (so labor overtime used), s_2 positive)

$2x_A + 5x_B + 10s_2$ (if s_1 positive, s_2 negative)

$2x_A + 5x_B + 15s_1 + 10s_2$ (if s_1, s_2 both negative)

Resolution

Write $s_1 = s_1^- - s_1^+$ and $s_=2 = s_2^- - s_2^+$, all $s_i^\pm \geq 0$

Interpretation:

s_1^- measures amount of unused labor
s_1^+ measures amount of overtime labor
s_1^- measures amount of unused machine time
s_1^+ measures amount of overtime on machines

The linear program in standard form:

Maximize $2x_A + 5x_B - 15s_1^+ - 10s_2^+$ (a linear objective)

subject to $2x_A + x_B + s_1^- - s_1^+ = 80$

$$x_A + 2x_B + s_2^- - s_2^+ = 65$$

where $x_A, x_B, s_1^-, s_1^+, s_2^-, s_2^+ \geq 0.$

Note there are feasible solutions with (say) $s_1^-, s_1^+ > 0$ (meaning unused labor *and* overtime). This is not realistic, but it is both intuitively and mathematically clear that this won't occur in an *optimal* solution.