Math 30210 — Introduction to Operations Research

Final Examination (120 points total)

Friday December 14, 2007, 4.15pm–6.15pm

Instructions: Please present your answers neatly and legibly. Complete all questions for full credit. You may not use notes or books. There are eight questions, each worth 15 points.

NAME: ____

Problem	Score	Problem	Score
1		**5**	
2		**6**	
3		**7**	
4		**8**	

Total	

1. A team of four workers, Alice, Bob, Charlie and Debbie, have to split three tasks between them (so one of the four will not perform a task). Alice cannot perform task I, and can perform tasks II and III at costs 3 and 2, respectively. Bob can perform tasks I, II and III at costs 4, 4 and 1 respectively, Charlie at costs 4, 2 and 3, respectively, and Debbie at costs 2, 5 and 4, respectively. Bob is the team captain, and must perform one of the tasks, but there is no cost associated with any of the others not performing a task. By introducing a dummy task with appropriate costs, set up the problem of finding a minimum-cost assignment of jobs to team members, and solve the problem using the Hungarian method. State clearly what the final assignment of workers to tasks is, and what the optimal cost is. For convenience, you may use the blank grids below.

2. Alice and Bob play the following game: each of them hides a coin in their fist, either a nickel, a dime or a quarter. At the same moment, they both reveal their hidden coins. If the hidden coins match, Alice gets Bob's coin and keeps hers; otherwise Bob get Alice's coin and keeps his.

   (a) Viewing this as a two-person zero-sum game, write down the payoff matrix in the grid below.

   (b) Explain why this is not a saddle-point game.

   (c) What is Alice's worst-case payoff if she chooses to play a nickel with probability .6, a dime with probability .3 and a quarter with probability .1?

   (d) Write down the linear programming problem that Alice should solve to determine her optimum mixed strategy.

3. Hi-V produces three types of juice drinks: Hi-X, Hi-Y and Hi-Z, using fresh strawberries, grapes and apples. The production data is encoded in the following table:

	Strawberry	Grape	Apple
Daily supply (in tonnes)	200	100	150
Cost per tonne	\$200	\$100	\$90
Litres of juice produced per tonne	1500	1200	1000
Juice ratios in Hi-X	1	0	1
Juice ratios in Hi-Y	1	1	2
Juice ratios in Hi-Z	0	2	3
Retail cost per litre bottle	\$1.15	\$1.25	\$1.20

Starting with variables C_X, C_Y and C_Z representing the daily number of cans of Hi-X, Hi-Y and Hi-Z that Hi-V produces, write down the linear programme that Hi-V needs to solve to maximize its daily revenue (sales revenue minus raw materials cost). (You may find it helpful to add more variables along the way).

4. A linear programming problem has been solved, and the following optimal tableau reached:

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln.
Max	1	0	0	6	3	2	1	0	46
x_7	0	0	0	4	3	4	2	1	7
x_1	0	1	0	2	1	-2	2	0	3
x_2	0	0	1	-1	0	-2	2	0	2

(a) It is then discovered that three additional constraints should have been considered:

$$x_1 + 2x_2 - 3x_3 \geq 3 \tag{1}$$
$$x_3 - 4x_5 + x_3 \leq 7 \tag{2}$$
$$5x_2 + 2x_4 - x_7 \leq 2 \tag{3}$$

Say which of these constraints effect the optimal solution already obtained, and which do not. Give a brief justification.

(b) The additional constraint $x_1 + x_2 + x_3 \leq 4$ definitely does effect the optimal solution. Using the blank tableau below, show how this constraint gets added into the final tableau and how the tableau should be modified in order to continue the search for the new optimum. Say whether the next iteration will be an iteration of the primal simplex algorithm or the dual simplex algorithm, and in either case identify departing and entering variables, and perform the next iteration of the algorithm (i.e., do the pivoting), and say (with justification) whether or not any more pivoting needs to be done in order to reach the new optimum.

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7		Soln.
Max	1	0	0	6	3	2	1	0		46
x_7	0	0	0	4	3	4	2	1		7
x_1	0	1	0	2	1	-2	2	0		3
x_2	0	0	1	-1	0	-2	2	0		2

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7		Soln.
Max	1	0	0	6	3	2	1	0		46
x_7	0	0	0	4	3	4	2	1		7
x_1	0	1	0	2	1	-2	2	0		3
x_2	0	0	1	-1	0	-2	2	0		2

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7		Soln.
Max										

5. (a) "The primal simplex method works by first finding any basic solution, and then moving to nearby better basic solutions until the optimal solution is found."

   The above statement is not quite completely correct. What's missing?

   (b) How can the phenomenon of degeneracy by spotted by looking at the simplex tableau?

   (c) What are the potential effects of degeneracy on the running of the simplex algorithm?

   (d) How can you tell, by looking at the optimum simplex tableau, whether there are alternative optima?

6. I have to pay a bill, and need to borrow money from some or all of my friends Alice, Bob and Carole. Alice is willing to give me up to \$100, subject to the condition that she provides no more than 50% of the total that I receive. Bob is willing to give me up to \$80, subject to the condition that he is not the only person who lends me money. Carole is willing to give me up to \$90, subject to the condition that I do not receive any money from Bob.

   (a) List two feasible alternatives for this problem.

   (b) List two infeasible alternatives for this problem.

   (c) If my objective is to maximize the amount of money I receive, what is my optimum feasible alternative? Justify!

7. Consider the following linear programming problem:

   Maximize $x_1 + 4x_2 + 6x_2$ subject to $x_1, x_2, x_3 \geq 0$ and

$$
\begin{aligned}
x_1 + 7x_2 - 3x_3 &\leq 16 && (4)\\
x_3 - 4x_5 + 8x_3 &\leq 18 && (5)\\
5x_2 + 2x_4 - x_7 &\leq 18. && (6)
\end{aligned}
$$

   (a) Find a feasible solution to this problem with all variables equal, and with the objective value as large as possible subject to this constraint.

   (b) Write down the dual problem.

   (c) Find a feasible solution to the dual problem with all variables equal, and with the dual objective value as small as possible subject to this constraint.

   (d) What can you say about the optimal value of the linear programming problem? Justify your answer.

8. (a) You are solving an integer programming problem using branch-and-bound:
Minimize $x_1 + 2x_2 + 3x_3$ subject to the constraints

$$\begin{aligned} 4x_1 + 3x_2 + 3x_3 &\leq 8 \\ x_i &\leq 1 \text{ for } i = 1, 2, 3 \end{aligned}$$

with $x_1, x_2, x_3 \geq 0$ and integral. You begin by solving without the integer constraint, and you discover that the optimum solution is given by $x_1 = .5$, $x_2 = 1$ and $x_3 = 1$. Explain what you would do in the next step of the algorithm.

(b) You are solving a traveling salesman problem using branch-and-bound. At some point in the process you have a large number of nodes that you still need to branch on. You pick one, branch on it, and in solving the resulting assignment problem you discover that one of the branches has as its solution a valid traveling salesman tour, with cost 45. Explain what you would do in the next step of the algorithm.

(c) The distances between five towns, imaginatively called A, B, C, D and E, are given in the diagram below (note that these towns are all in New Jersey, so distance is not commutative).

	A	B	C	D	E
A	X	3	5	4	9
B	3	X	6	4	8
C	4	5	X	2	7
D	3	8	1	X	2
E	2	6	7	1	X

Use the greedy algorithm to find a traveling salesman tour that starts and ends at A, calculate its cost, and improve the cost of the tour by doing 2-city subtour reversing.

A pair of silly bonus questions (5 points each), in case you have the time, energy and/or interest:

1. The drill sergeant orders her 24 troops to arrange themselves into a 4 by 6 rectangle. She asks the tallest soldier in each row to raise her right hand, and notes that Alice is the shortest person with her right hand raised. She then asks the shortest person in each column to raise her left hand, and notes that Private Anders is the tallest person with her left hand raised. Are Alice and Private Anders necessarily the same person? If not, what can you say about the relative heights of Alice and Private Anders?

2. A magic square is an arrangement of the numbers 1 through 9 in a 3 by 3 grid in such a way that the numbers in each row and the numbers in each column have the same sum (necessarily 15). Write the problem of finding a magic square as an integer programming problem. (**Hint**: it comes down to finding a way to encode the constraint $x \neq y$ in a linear way. Once you see how to do this, is not too hard to encode a Sudoku puzzle as a integer programming problem, as well!)