Math 10860, spring 2020

First midterm exam, Monday March 2

Solutions

1. (4+4+4 points)

   (a) State the first part of the Fundamental Theorem of Calculus (FTOC1) (with all necessary hypotheses).

   Solution: Let f be a function defined on an interval I, that is integrable on I (meaning, but there isn't need to say this: f is integrable on every finite closed interval contained in I). Let a be any point in I, and let the function $F : I \to \mathbb{R}$ be defined by

   $$F(x) = \int_a^x f(t)dt.$$

   If f is continuous at a point $c \in I$ then F is differentiable at c, and $F'(c) = f(c)$. **Note**: Not correct to replace this with the weaker "If f is continuous then F is differentiable, and $F' = f$".

   Here's another acceptable way to present the statement: Let $f : [a, b] \to \mathbb{R}$ be integrable. Let the function $F : [a, b] \to \mathbb{R}$ be defined by

   $$F(x) = \int_a^x f(t)dt.$$

   If f is continuous at a point $c \in [a, b]$ then F is differentiable at c, and $F'(c) = f(c)$.

   (b) Use FTOC1 to prove that if f is a continuous function defined on an interval I then there are functions g defined on I for which $g' = f$, and that for any such g and any $a < b$ in I

   $$\int_a^b f = g(b) - g(a).$$

   Solution: First, the existence of such functions: since f is continuous on I it is integrable on I (we proved that all continuous functions are integrable), and so we can use FTOC1 (again using that f is continuous everywhere) to conclude that the function F defined in the previous part satisfies $F' = f$. For the rest of the question: suppose g defined on I satisfies $g' = f$. Since also $F' = f$ we know that g and F differ on I by a universal constant, C say: for all $x \in I$

   $$F(x) = g(x) + C.$$

   In particular $F(a) = g(a) + C$, so $C = F(a) - g(a)$. But $F(a) = \int_a^a f = 0$, so $C = -g(a)$, and

   $$F(x) = g(x) - g(a).$$

   Evaluating at $x = b$ and using $F(b) = \int_a^b f$ we get $\int_a^b f = g(b) - g(a)$.

   Note: This question was asking you to deduce a *weak* form FTOC2 (assuming continuity of f in place of the weaker integrability) from FTOC1, rather than proving the strong form of FTOC2 from first principles!

(c) State the second part of the Fundamental Theorem of Calculus (with all necessary hypotheses).

Solution: Suppose that $f : [a, b] \to \mathbb{R}$ is integrable, and also that there is a function $g : [a, b] \to \mathbb{R}$ satisfying $g' = f$. Then

$$\int_a^b f = g(b) - g(a).$$

2. (4+5 points)

(a) A function $f : (a, b) \to \mathbb{R}$ is bounded on every closed interval contained in (a, b), but is unbounded on (a, b), and in fact $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = \infty$. What is the correct way to interpret the improper integral $\int_a^b f$? (Your answer should address when the improper integral exists).

Solution: Let c be any number in (a, b). The improper integral $\int_a^b f$ exists exactly when each of the limits

$$\lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{c} f$$

and

$$\lim_{\varepsilon\to 0^-} \int_{c}^{b+\varepsilon} f$$

exist, and in this case

$$\int_a^b f = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{c} f + \lim_{\varepsilon\to 0^-} \int_{c}^{b+\varepsilon} f.$$

(b) Find all real numbers r for which $\int_0^\infty \frac{dx}{(1+x)^r}$ exists (with justification; you can state without proof any (correct) properties of any functions that you use in your justification).

Solution: For all r, the function $f_r(x) = 1/(1 + x)^r$ is bounded and integrable on every interval of the form $[0, N]$, $N > 0$, so the correct interpretation of the improper integral is

$$\int_0^\infty \frac{dx}{(1+x)^r} = \lim_{N\to\infty} \int_0^N \frac{dx}{(1+x)^r}.$$

Note: There was no need to consider $\int_0^1$ and $\int_1^\infty$ separately here, because the function being integrated is bounded near 0.

For $r = 1$ we have

$$\lim_{N\to\infty} \int_0^N \frac{dx}{(1+x)} = \lim_{N\to\infty} [\log(x+1)]_0^N = \lim_{N\to\infty} \log N = \infty.$$

For $r \neq 1$ we have

$$\lim_{N\to\infty} \int_0^N \frac{dx}{(1+x)^r} = \lim_{N\to\infty} \left[\frac{1}{(1-r)(1+x)^{r-1}}\right]_0^N = \lim_{N\to\infty} \left(\frac{1}{(1-r)(1+N)^{r-1}} - \frac{1}{1-r}\right).$$

For $r > 1$ the power in the denominator is > 0 and so the limit exists (and is $1/(r - 1)$). For $r < 1$ the power is negative, so the limit is infinite.
In conclusion, the improper integral exists if $r > 1$ and not if $r \leq 1$.

3. (4+5 points)

(a) Give the definition of the function cos on the domain $[0, 2\pi]$.

Solution: For $x \in [0, \pi]$, we define $\cos(x)$ to be $A^{-1}(x/2)$ where

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\, dt.$$

For $x \in [\pi, 2\pi]$ we define $\cos(x)$ to be $\cos(2\pi - x)$.

(b) State the domain and range of arccos (the function also known as $\cos^{-1}$) and compute its derivative. You may assume any facts you know about sin, cos. You should state any facts you use about derivatives of inverse functions. Your final answer should not involve any trigonometric functions.

Solution: The domain of arccos is $[-1, 1]$ and its range is $[0, \pi]$.
Since $\arccos(x)$ is the inverse of cos (restricted to the domain $[0, \pi]$), and for any function f we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

as long as f is differentiable at $f^{-1}(x)$ with non-zero derivative, we get that

$$\begin{aligned}(\arccos)'(x) &= \frac{1}{\cos'(\arccos(x))} \\ &= \frac{-1}{\sin(\arccos(x))},\end{aligned}$$

as long as $\sin(\arccos(x)) \neq 0$, that is, as long as $x \neq -1, 1$ (at which points arccos is π or 0, both places where sin is 0).
To re-express $\sin(\arccos(x))$ without mentioning trigonometric functions, note that

$$\sin^2(\arccos(x)) + \cos^2(\arccos(x)) = 1 \quad \text{or} \quad \sin(\arccos(x)) = \pm\sqrt{1 - x^2}.$$

Noting that $\arccos(x)$ takes values between 0 and π, where sin is non-negative, we see that we should take the positive square root: $\sin(\arccos(x)) = +\sqrt{1 - x^2}$.
In summary,

$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}}$$

as long as $x \in (-1, 1)$.

4. (5+5 points) (**NB**: these two parts are not intended to be related)

(a) Find, with proof, all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f' = -f$. (**Note**: $f' = -f$, not $f' = f$.)

Solution: We

Claim: the functions which satisfy $f' = -f$ are exactly the functions $f(x) = ce^{-x}$, where c is an arbitrary constant.

To prove this, let f be a function satisfying $f' = -f$, and consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = e^x f(x)$. We have

$$g'(x) = e^x f'(x) + e^x f(x) = e^x(f'(x) + f(x)) = 0$$

for all x (using $f' = -f$), so that $g(x) = c$ for all x, for some constant c, and so (using $e^x \neq 0$ for any x)

$$f(x) = ce^{-x}.$$

This shows that the functions which satisfy $f' = -f$ have the form $f(x) = ce^{-x}$, where c is an arbitrary constant.

(b) Using what you know about functions satisfying $f'' + f = 0$ (or otherwise, but in this case not for full credit) prove that sin is an odd function.

Solution: One thing we know (from a theorem in class) is that if $f'' + f = 0$, $f(0) = 0$ and $f'(0) = 0$, then $f(x) = 0$ for all x.
Consider the function $f(x) = \sin(-x) + \sin(x)$. It is easy to check that $f'' + f = 0$, $f(0) = 0$ and $f'(0) = 0$, so that $f(x) = 0$ for all x, or

$$\sin(-x) = -\sin(x)$$

for all x, i.e., sin is an odd function.

5. (Bonus question, 2 points) What is $\sin(\pi/8)$?

Solution: Here is one approach. We have

$$\frac{\sqrt{2}}{2} = \cos(\pi/4) = \cos^2(\pi/8) - \sin^2(\pi/8) = 1 - 2\sin^2(\pi/8),$$

so, noting that $\sin(\pi/8) > 0$ (since $\pi/8 \in (0, \pi)$),

$$\sin(\pi/8) = +\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$