Math 10860, Honors Calculus 2

Homework 6 **NAME**:

Solutions

1. Consider the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

   (a) Verify that f is continuous at 0. (We did this informally in the fall, now we can do it formally. This part should be trivial.)

   Solution: There are two possible valid approaches:

   - use L'Hôpital's rule, or:
   - use what we now know about the derivative of sin:

   $$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin(0+x) - \sin 0}{x} = \sin' 0 = \cos 0 = 1.$$

   (b) Verify that f is differentiable at 0, and find $f'(0)$. (This requires a small ε-δ argument; it's the argument we skipped in class, when we talked about sin and cos being differentiable at $0, \pm\pi, \pm 2\pi$, et cetera. Basically what you have to show, either in general or for this specific example, that if f is defined on some open interval that contains a, f is continuous, f is differentiable everywhere except (possibly) at a, and if there is some number L such that f' approaches L near a, then in fact f is differentiable at a, and the derivative there is L).

   Solution: Away from 0, f is differentiable with derivative $\frac{x\cos x - \sin x}{x^2}$. But what about at 0?
   An application of L'Hôpital's rule yields that

   $$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} = \lim_{x\to 0} \frac{-x\sin x}{2x} = 0.$$

   In other words, the function f' (which we now know to be defined everywhere except perhaps at 0), approaches 0 near 0. But also, as observed in part (a), the function f is continuous at 0. We now need the following technical result, alluded to in the question:

   > If a function f is continuous at a, f' exists near a, and $\lim_{x\to a} f'(x)$ exists and equals L then $f'(a) = L$.

This allows us to conclude that $f'(0)$ exists and equals 0.

There are two ways to prove the quoted result above. The "hard" way is to give a (fairly short) ε-δ proof. I won't reproduce it here; it is Lemma 12.6 of the course notes.

The "soft" way is through the most general form of L'Hôpital's rule. We are looking at

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

This expression doesn't satisfy the conditions of the easily proved, weak, form of L'Hôpital's rule (Claim 9.11 in the notes), because the function in the numerator (sending b to $f(b) - f(a)$) is not known to be differentiable at a. However, it does satisfy all the conditions of the strong (kick-ass) form of L'Hôpital's rule (Theorem 9.12 of the notes). Specifically:

- $\lim_{b \to a} f(b) - f(a) = 0$
- $\lim_{b \to a} b - a = 0$
- $$\lim_{b \to a} \frac{\text{deriv. of num. wrt } b}{\text{deriv. of denom. wrt } b} = \lim_{b \to a} \frac{f'(b)}{1} = L.$$

  So Theorem 9.12 allows us to conclude that

  $$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = L.$$

  The second approach was "soft" because it just used a known theorem; but remember that the full version of L'Hôpital's rule (and that full version *is* needed here) is quite hard to prove — we skipped it in the fall because it would take too long. On the other hand, the ε-δ proof of Lemma 12.6 is quite easy! The second approach here is an example of using a sledgehammer to crack a nut.

2. Compute

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right).$$

Solution: We have

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}.$$

By L'Hôpital's rule,

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \to 0} \frac{\sin x - x}{x \sin x} \\ &= \lim_{x \to 0} \frac{\cos x - 1}{x \cos x + \sin x} \\ &= \lim_{x \to 0} \frac{-\sin x}{-x \sin x + 2 \cos x} \\ &= 0. \end{aligned}$$

3. This question is about Machin's formula for π.

(a) Prove that for all α and β for which all of $\tan(\alpha+\beta)$, $\tan\alpha$ and $\tan\beta$ exist,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Solution: Assuming $\tan(\alpha+\beta)$ exists we have

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

As long as both $\tan\alpha$, $\tan\beta$ exist, it is legitimate to divide through by $\cos\alpha\cos\beta$ to get

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

as required.

(b) Deduce from the previous part that for certain x, y,

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right).$$

Determine exactly what conditions on x, y make this identity valid.

Solution: For all reals x, y, there are numbers α, β with $\tan\alpha = x$ and $\tan\beta = y$, and with $-\pi/2 < \alpha, \beta < \pi/2$. It is tempting now to say:

"As long as $\alpha + \beta \neq \pm\pi/2$, we know

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

so

$$\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}$$

so

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)."$$

This is correct up to the last line. To deduce that

$$\arctan\tan(\arctan x + \arctan y) = \arctan x + \arctan y$$

we need to know that $\arctan x + \arctan y$ is in the part of the domain of tan on which we defined arctan, namely the restricted domain $(-\pi/2, \pi/2)$. Once we know this, everything is ok.

So the assumption we need to make on x and y to conclude

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

is that

$$-\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$$

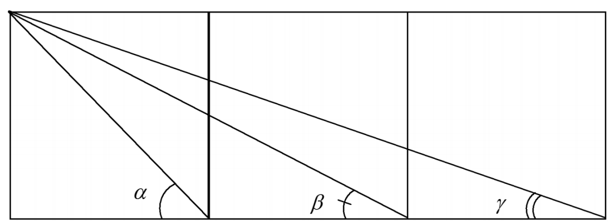
(which also ensures that $\alpha + \beta \neq \pm\pi/2$, and therefore allows the application of the tangent summation formula).

This is an awkward condition to check. Since we'll need the formula later in the question, here is an easy-to-check condition that implies $-\pi/2 < \arctan x + \arctan y < \pi/2$, and applies in all cases where will need it to:

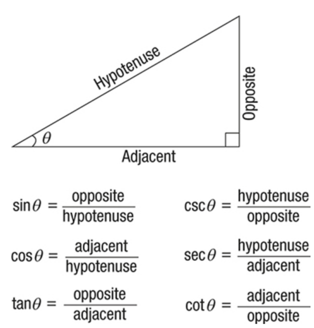
$$x, y \in [-1, 1], \quad \text{and not both} = 1 \text{ or} = -1.$$

(Since on $[-1, 1]$, arctan increases from $-\pi/4$ to $\pi/4$, this is equivalent to $\arctan x, \arctan y \in [-\pi/4, \pi/4]$ and not both $= -\pi/4$ or $= \pi/4$; this clearly gives $-\pi/2 < \arctan x + \arctan y < \pi/2$.)

There is in fact a simple-to-check condition on x, y that is necessary and sufficient for $-\pi/2 < \arctan x + \arctan y < \pi/2$, namely $xy < 1$, but we won't prove that here.



(c) (This part is an aside, but hopefully a cute one). The picture above shows a 1 by 3 rectangle divided into three 1 by 1 squares. Show that $\alpha = \beta + \gamma$. You may assume the connection between our analytic trigonometric functions, and the ratios of sides of right-angled triangles, as illustrated below:



Solution: We have $\alpha = \arctan 1$, $\beta = \arctan 1/2$ and $\gamma = \arctan 1/3$. Since $1/2, 1/3 \in [-1, 1]$,, and not both $= 1$ or $= 1$, the arctangent summation formula applies[1] and says

$$\beta + \gamma = \arctan(1/2) + \arctan(1/3) = \arctan\left(\frac{1/2 + 1/3}{1 - (1/2)(1/3)}\right) = \arctan(1) = \alpha.$$

[1]Or, more directly, it is clear that $\pi/4 = \alpha > \beta > \gamma > 0$, so $0 < \alpha + \beta < \pi/2$, so arctangent summation formula applies.

One could also approach this problem using arccos and/or arcsin, but at some point, no matter how the problem is approached, one has to address the issue that

Inverse trig function(trig function(x))= x only for x in a certain range.

(d) Prove Machin's formula:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

Solution: Machin's formula is equivalent to

$$\arctan 1 + \arctan\frac{1}{239} = \left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right) + \left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right).$$

By the arctangent summation formula (valid in all cases, either by our sufficient criterion, or by observing that $\arctan x \in (0, \pi/4)$ for all $0 < x < 1$, so clearly every time we look at $\arctan x + \arctan y$ below, it is in the range $(0, \pi/2)$),

$$\arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\frac{5}{12}$$

and so the right-hand side above is

$$\arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\frac{120}{119},$$

while the left-hand side above is

$$\arctan 1 + \arctan\frac{1}{239} = \arctan\frac{120}{119}$$

also.

4. (a) Find formulae for $\sin 3x$ in terms of $\sin x$, and for $\cos 3x$ in terms of $\cos x$.

Solution: We have

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \\
&= 3\sin x \cos^2 x - \sin^3 x \\
&= 3\sin x(1 - \sin^2 x) - \sin^3 x \\
&= 3\sin x - 4\sin^3 x.
\end{aligned}$$

and

$$\begin{aligned}
\cos 3x &= \cos(2x + x) \\
&= \cos 2x \cos x - \sin 2x \sin x \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x \\
&= \cos^3 x - 3(1 - \cos^2 x)\cos x \\
&= 4\cos^3 x - 3\cos x.
\end{aligned}$$

(b) Deduce that (not unexpectedly) $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Solution: From the formula for $\sin 3x$ in terms of $\sin x$, evaluated at $x = \pi/6$, we get

$$4\sin^3(\pi/6) - 3\sin(\pi/6) + 1 = 0.$$

So $\sin(\pi/6)$ is a solution to the polynomial equation $f(x) = 0$, where $f(x) = 4x^3 - 3x + 1$, and must lie between 0 and 1.

Now $f(0) = 1$ and $f(1) = 2$, and $f'(x) = 12x^2 - 3$, which is negative on $[0, 1/2)$ and positive on $(1/2, 1]$. It follows that f decreases from 1 to $f(1/2)$ on $[0, 1/2]$ and increases from $f(1/2)$ to 2 on $[1/2, 1]$. We have $f(1/2) = 0$, so $f > 0$ on $[0, 1/2) \cup (1/2, 1]$. Hence $x = 1/2$ is the unique solution to $f(x) = 1/2$ on $[0, 1]$, and $\sin(\pi/6) = 1/2$ as claimed.

From the formula for $\cos 3x$ in terms of $\sin x$, evaluated at $x = \pi/6$, we get

$$4\cos^3(\pi/6) - 3\cos(\pi/6) = 0.$$

So $\cos(\pi/6)$ is a solution to the polynomial equation $g(x) = 0$, where $g(x) = 4x^3 - 3x$, and must lie (strictly) between 0 and 1. It is easy to check that the unique such solution is $x = \sqrt{3}/2$, so $\cos(\pi/6) = \sqrt{3}/2$ as claimed.

5. If $t = \tan(x/2)$ find simple expressions for $\sin x$ and $\cos x$ in terms of t. ("Simple" here means that the expressions should be rational functions in t. The only assumption you should make on x is that $\tan(x/2)$ is defined.)

Solution 1 (a weak solution): We have $t = \tan(x/2)$, so, assuming $-\pi < x < \pi$, we get $x = 2\arctan t$. It follows that

$$\sin x = \sin(2(\arctan t)) = 2\sin(\arctan t)\cos(\arctan t).$$

Now

$$t = \tan(x/2) = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)}{\sqrt{1 - \sin^2(x/2)}}$$

(note that we take the positive square root, since $-\pi < x < \pi$ implies $\cos(x/2) > 0$). Rearranging, we get

$$\sin^2(x/2) = \frac{t^2}{1 + t^2}$$

or

$$\sin(\arctan t) = \sin(x/2) = \frac{t}{\sqrt{1 + t^2}}$$

(note that if $x \geq 0$ then $t \geq 0$, so the above expression is positive, as it should be since $\sin(x/2) \geq 0$ in this case; and if $x \leq 0$ then $t \leq 0$, so the above expression is negative, as it should be since $\sin(x/2) \leq 0$ in this case).

Since $\cos(x/2) = \sqrt{1 - \sin^2(x/2)}$ in our range of x (on which $\cos(x/2)$ is non-negative), we get also

$$\cos(\arctan t) = \cos(x/2) = \sqrt{1 - \frac{t^2}{1 + t^2}} = \frac{1}{\sqrt{1 + t^2}}.$$

So,

$$\sin x = \frac{2t}{1+t^2}.$$

Also,

$$\cos x = \cos(2(\arctan t)) = \cos^2(\arctan t) - \sin^2(\arctan t) = \frac{1-t^2}{1+t^2}.$$

The issue with this solution is the assumption that $-\pi < x < \pi$, which is not actually required. For example, if $x = 7\pi/6$ then $x/2 = 7\pi/12$ and $t = -2 - \sqrt{3}$, so

$$\frac{2t}{1+t^2} = -\frac{1}{2} = \sin(7\pi/6)$$

and

$$\frac{1-t^2}{1+t^2} = -\frac{\sqrt{3}}{2} = \cos(7\pi/6).$$

Solution 2 (a much better solution): Here's an alternate solution, that does not make any assumption on x, other than that $x/2$ is in the domain of the tangent function.

Knowing $t = \tan(x/2)$, we have

$$\begin{aligned}
\sin x &= 2\sin(x/2)\cos(x/2) \\
&= 2\frac{\sin(x/2)\cos^2(x/2)}{\cos(x/2)} \\
&= 2\tan(x/2)\cos^2(x/2) \\
&= \frac{2t}{\sec^2(x/2)} \\
&= \frac{2t}{1+\tan^2(x/2)} \\
&= \frac{2t}{1+t^2},
\end{aligned}$$

with everything valid exactly as long as $\tan(x/2)$ is defined.

Also, since $\cos x = \cos^2(x/2) - \sin^2(x/2)$ and $1 = \cos^2(x/2) + \sin^2(x/2)$, we have

$$\begin{aligned}
\cos x &= 1 - 2\sin^2(x/2) \\
&= 1 - 2\frac{\sin^2(x/2)\cos^2(x/2)}{\cos^2(x/2)} \\
&= 1 - 2\tan^2(x/2)\cos^2(x/2) \\
&= 1 - 2t^2\cos^2(x/2) \\
&= 1 - 2\frac{t^2}{\sec^2(x/2)} \\
&= 1 - 2\frac{t^2}{1+\tan^2(x/2)} \\
&= 1 - 2\frac{t^2}{1+t^2} \\
&= \frac{1-t^2}{1+t^2},
\end{aligned}$$

again with everything valid exactly as long as $\tan(x/2)$ is defined.

6. This question is about the *hyperbolic trigonometric functions* sinh and cosh, defined as follows:

$$\begin{array}{rcl} \sinh : \mathbb{R} \to \mathbb{R} & \quad\text{and}\quad & \cosh : \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{e^x - e^{-x}}{2} & & x \mapsto \frac{e^x + e^{-x}}{2}. \end{array}$$

It is a long question, and I don't expect you to do all of it carefully. You *should* do every part in a less careful, scratch-work sort of way (in particular, you will do your soul good if you verify Part (h)), but here are the only parts that you should turn in for grading:

- Part (c)
- Parts (d)(iii) and (d)(vii)
- Parts (e), (f) and (g).

(a) Look at a graph of sinh. Check that all the features of the graph follow from basic calculus considerations: sinh is increasing from $-\infty$ (at $-\infty$) to ∞ (at ∞), never has zero derivative, is concave for negative x, convex for positive x, and passes through the origin.

Solution: Routine (I hope!).

(b) Look at a graph of cosh. Check that all the features of the graph follow from basic calculus considerations: cosh decreases from ∞ (at $-\infty$) to 1 (at 0), then increases to ∞ (at ∞), and is always convex.

Solution: Routine.

(c) Name the famous monument, located in the midwest, whose shape is an upside-down cosh graph[2].

Solution: It's the Jefferson National Expansion Memorial (a.k.a. the Gateway Arch) in St. Louis, MO.

(d) Check that sinh and cosh satisfy the following identities, that are very reminiscent of identities satisfied by sin and cos:

i. $\cosh^2 - \sinh^2 = 1$.

Solution: Routine.

ii. $\tanh^2 + 1/\cosh^2 = 1$ (Here tanh is defined to be $\sinh/\cosh$; note that its domain is all reals).

Solution: Routine.

iii. $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$

Solution:

$$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2},$$

while $\sinh x \cosh y + \sinh y \cosh x$

$$\begin{aligned}
&= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^y - e^{-y}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{y+x} + e^{y-x} - e^{-y+x} - e^{-y-x}}{4} \\
&= \frac{2e^{x+y} - 2e^{-x-y}}{4} \\
&= \frac{e^{x+y} - e^{-x-y}}{2},
\end{aligned}$$

so the two sides are equal.

iv. $\cosh(x+y) = \cosh x \cosh y + \sinh y \sinh x$.

Solution: Routine (same as last part).

v. $\sinh' = \cosh$.

Solution: Routine.

vi. $\cosh' = \sinh$.

Solution: Routine.

[2]This is not some math professor BS. See, for example, the "Mathematical elements" section of its Wikipedia page. (You can also find the verification on its official page, but Wikipedia presents it better.)

vii. $\tanh' = 1/\cosh^2$

Solution: Since

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

have

$$\begin{aligned}\tanh'(x) &= \frac{\sinh'(x)\cosh(x) - \sinh(x)\cosh'(x)}{\cosh^2(x)} \\ &= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} \\ &= \frac{1}{\cosh^2(x)},\end{aligned}$$

using parts (i), (v) and (vi).
One could also directly differentiate

$$\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}}$$

with respect to x and check that one gets

$$\frac{1}{\left(\frac{e^x+e^{-x}}{2}\right)^2}.$$

(e) sinh is invertible, with inverse $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$. Because sinh never has zero derivative, $\sinh^{-1}$ is differentiable everywhere. Find a very simple expression for $(\sinh^{-1})'(x)$ (one that does not involve hyperbolic trigonometric functions).

Solution: We have

$$\begin{aligned}(\sinh^{-1})'(x) &= \frac{1}{\sinh'(\sinh^{-1}(x))} \\ &= \frac{1}{\cosh(\sinh^{-1}(x))}.\end{aligned}$$

Now

$$\cosh^2(\sinh^{-1}(x)) - \sinh^2(\sinh^{-1}(x)) = 1$$

(part (d)(i)), so

$$\cosh^2(\sinh^{-1}(x)) = x^2 + 1,$$

and so, since cosh is always positive,

$$\cosh(\sinh^{-1}(x)) = \sqrt{x^2+1}$$

leading finally to

$$(\sinh^{-1})'(x) = \frac{1}{\sqrt{x^2+1}}.$$

(f) Verify that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ for all real x.

Solution: Because sinh is an invertible function whose domain and range is all reals, by definition $\sinh^{-1}(x)$ is that unique real number y such that $\sinh(y) = x$. To check that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$, we therefor need only check that $\sinh(\log(x + \sqrt{x^2+1})) = x$. And indeed, we have

$$\begin{aligned}
\sinh(\log(x + \sqrt{x^2+1})) &= \frac{e^{\log(x+\sqrt{x^2+1})} - e^{-\log(x+\sqrt{x^2+1})}}{2} \\
&= \frac{x + \sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}}}{2} \\
&= \frac{(x+\sqrt{x^2+1})^2 - 1}{2(x+\sqrt{x^2+1})} \\
&= \frac{2x^2 + 2x\sqrt{x^2+1}}{2(x+\sqrt{x^2+1})} \\
&= x.
\end{aligned}$$

(g) Calculate $\int_a^b \frac{dt}{\sqrt{1+t^2}}$.

Solution: We have just learned that $\sinh^{-1}(t)$ is an antiderivative of $1/\sqrt{1+t^2}$, and that $\log(t + \sqrt{t^2+1})$ is another expression for $\sinh^{-1}(t)$. So by FTOC part 2,

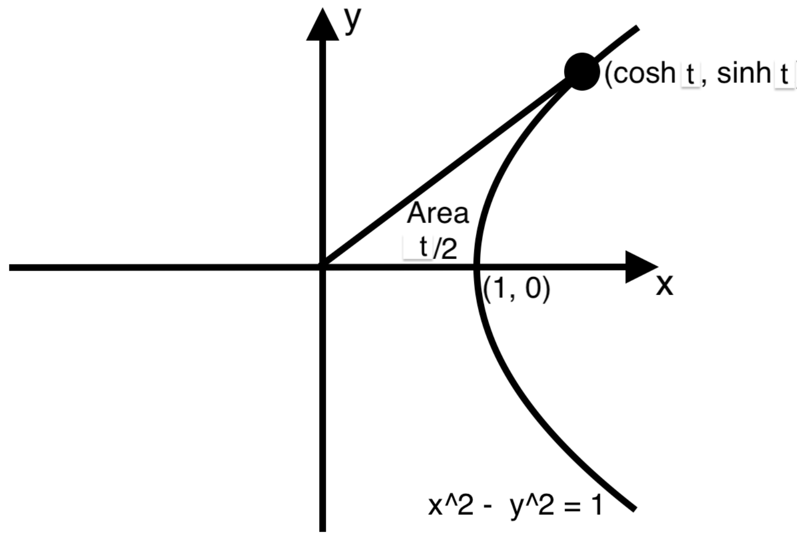
$$\begin{aligned}
\int_a^b \frac{dt}{\sqrt{1+t^2}} &= \log(b + \sqrt{b^2+1}) - \log(a + \sqrt{a^2+1}) \\
&= \log\left(\frac{b + \sqrt{b^2+1}}{a + \sqrt{a^2+1}}\right).
\end{aligned}$$

(h) This question justifies the name "hyperbolic trigonometric functions" for sinh and cosh.

Consider the curve in the (x, y)-plane consisting of all points (x, y) satisfying $x^2 - y^2 = 1$ (this curve is called a *hyperbola*). Let $P = (a, b)$ be a point on the curve, with $a \geq 1$ and $b \geq 0$. Suppose that the area A bounded by

- the x-axis between $(1, 0)$ and $(0, 0)$,
- the line segment from $(0, 0)$ to P, and
- the curve $x^2 - y^2 = 1$ between P and $(1, 0)$

is $t/2$ (see the picture below). Prove that $a = \cosh t$ and $b = \sinh t$. (So: the *hyperbolic* trigonometric functions can be defined in **exact** analogy with the ordinary trigonometric functions, by replacing the circle $x^2 + y^2 = 1$ with the *hyperbola* $x^2 - y^2 = 1$).



Solution: Not routine at all! The solution to this exercise is described in Section 12.5 of the course notes.