Math 10860, Honors Calculus 2

Homework 4 **NAME**:

Solutions

1. Decide whether or not the following improper integrals exist.

   (a) $\int_0^\infty \frac{dx}{\sqrt{1+x^3}}$.

   Solution: Certainly $\int_0^N \frac{dx}{\sqrt{1+x^3}}$ exists for all $N \geq 0$ (integrand is bounded on those intervals). In particular, $\int_0^1 \frac{dx}{\sqrt{1+x^3}}$ exists, so that leaves us with considering $\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$.
   We have
   $$0 \leq \frac{1}{\sqrt{1+x^3}} \leq \frac{1}{\sqrt{x^3}}$$
   for all $x \geq 1$. Since $\int_1^\infty \frac{1}{\sqrt{x^3}}$ exists ($\star$), from the comparison result proved in class we get that $\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$ exists, and so $\int_0^\infty \frac{dx}{\sqrt{1+x^3}}$ exists.
   ($\star$) Why? Since the derivative of $-2x^{-1/2}$ is $\frac{1}{\sqrt{x^3}}$, by FTOC we have
   $$\int_1^N \frac{dx}{\sqrt{x^3}} = -2N^{-1/2} + 2 \times 1^{-1/2} = 2 - \frac{2}{\sqrt{N}} \to 2 \quad \text{as } N \to \infty.$$

   (b) $\int_0^\infty \frac{dx}{x\sqrt{1+x}}$.

   Solution: Because $1/(x\sqrt{1+x})$ is unbounded near 0, the integral exists iff both $\int_0^1 \frac{dx}{x\sqrt{1+x}}$ and $\int_1^\infty \frac{dx}{x\sqrt{1+x}}$ exist.
   There is no problem with $\int_1^\infty \frac{dx}{x\sqrt{1+x}}$. However, on the interval $[\varepsilon, 1]$ (for any $\varepsilon > 0$) we have
   $$\frac{\sqrt{2}}{x\sqrt{1+x}} \geq \frac{1}{x} \geq 0$$
   so
   $$\int_\varepsilon^1 \frac{dx}{x\sqrt{1+x}} \geq \int_\varepsilon^1 \frac{dx}{x}.$$
   Mimicing a proof we saw in class, the latter integral can be made arbitrarily large by choosing ε small enough (in particular, $\int_{1/2^n}^1 \frac{dx}{x} \geq n/2$), and so also $\int_\varepsilon^1 \frac{dx}{x\sqrt{1+x}}$ can be made arbitrarily large.
   Conclusion: $\int_0^1 \frac{dx}{x\sqrt{1+x}}$ doesn't exist, and so neither does $\int_0^\infty \frac{dx}{x\sqrt{1+x}}$.

2. Suppose $\int_{-\infty}^{\infty} f$ exists. Let h, g be functions with $h(N) \to -\infty$ and $g(N) \to +\infty$ as $N \to +\infty$. Prove that

$$\lim_{N\to\infty} \int_{h(N)}^{g(N)} f$$

exists and equals $\int_{-\infty}^{\infty} f$.

Solution: Give $\varepsilon > 0$ there is $n_0 > 0$ such that for all $N > n_0$ we have

$$\int_0^N f \in \left(\int_0^\infty f - \varepsilon, \int_0^\infty f + \varepsilon\right).$$

Because $g(N) \to \infty$ as $N \to \infty$, there is $m > 0$ such that for all $N > m$, $g(N) > n_0$, and so

$$\int_0^{g(N)} f \in \left(\int_0^\infty f - \varepsilon, \int_0^\infty f + \varepsilon\right).$$

Since this was true for arbitrary $\varepsilon > 0$, we conclude that

$$\int_0^{g(N)} f \to \int_0^\infty f \quad \text{as} \quad N \to \infty.$$

By a similar argument we get

$$\int_{h(N)}^0 f \to \int_{-\infty}^0 f \quad \text{as} \quad N \to \infty.$$

These two facts together show that $\lim_{N\to\infty} \int_{h(N)}^{g(N)} f$ exists and equals $\int_{-\infty}^{\infty} f$.

3. Check that each of the following functions f is actually invertible, and find (a fairly simple expression for) f^{-1} for each. Specify the domain and range of f^{-1} in each case.

   (a) $f(x) = x^3 + 1$.

   Solution: f is increasing on $\mathbb{R}$, so is invertible. f clearly has range $\mathbb{R}$, so both the domain and range of inverse f^{-1} is $\mathbb{R}$. Evidently f^{-1} is given by

   $$f^{-1}(x) = \sqrt[3]{x-1}.$$

   (b) $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$

   Solution: f neither increasing nor decreasing on $\mathbb{R}$. However, it is injective: if $x \neq y$, then either

   - $x, y \in \mathbb{Q}$, in which case $f(x) \neq f(y)$ (since $f(x) = x$, $f(y) = y$), or
   - $x, y \notin \mathbb{Q}$, in which case $f(x) \neq f(y)$ (since $f(x) = -x$, $f(y) = -y$), or
   - one of x, y is rational, the other irrational, in which case $f(x) \neq f(y)$ (since one of x, y is rational, the other irrational).

So f is invertible. The range of f is all of $\mathbb{R}$, so both the domain and range of inverse f^{-1} is $\mathbb{R}$. Evidently f^{-1} is just f itself.

(c) $f(x) = x + [x]$. (Remember that $[x]$ is the largest integer less than or equal to x.)

Solution: f is increasing on $\mathbb{R}$, so is invertible (and the range of the inverse is $\mathbb{R}$). It is continuous on the intervals of the form $[n, n+1)$, $n \in \mathbb{Z}$, where it takes on the range of values $[2n, 2n+1)$. So the range of f (and hence the domain of f^{-1}) is $\cup_{n\in\mathbb{Z}}[2n, 2n+1)$.

On the interval $[2n, 2n+1)$, the inverse increases linearly along $[n, n+1)$, so the inverse can be expressed as follows:

$$f^{-1}(x) = x - n \quad \text{if } x \in [2n, 2n+1) \text{ for some } n \in \mathbb{Z}.$$

(This may be expressible compactly in terms of the floor function $[x]$, but I didn't think about it).

(d) $f(x) = \frac{x}{1-x^2}$, $-1 < x < 1$.

Solution: f is increasing on $(-1, 1)$ (it is differentiable, with derivative $x^2 + 1/(x^2-1)^2$, which is always positive). Thinking about limits as x approaches -1 from above and 1 from below, we get that the range is $\mathbb{R}$, So f is invertible with domain of f^{-1} being $\mathbb{R}$, range $(-1, 1)$.

To find an expression for $f^{-1}(x)$, set (for convenience) $f^{-1}(x) = y$. We have $f(y) = x$, so $y/(1-y^2) = x$ or $xy^2 + y - x$, or

$$y = \frac{-1 \pm \sqrt{1+4x^2}}{2x}.$$

Which of these two candidates for f^{-1} is the right one? We know, for example, that $f(1/2) = 2/3$, so $f^{-1}(2/3) = 1/2$. Now

$$\frac{-1 \pm \sqrt{1+4(2/3)^2}}{2(2/3)} = \frac{-3 \pm 5}{4}.$$

By taking the "+" in $\pm$ we get $1/2$, but by taking the "$-$" we get -2; so we should take the "+".

It's tempting to now say "The inverse of f is given by

$$f^{-1}(x) = \frac{-1 + \sqrt{1+4x^2}}{2x}."$$

Unfortunately, this expression is not defined at $x = 0$, while the inverse is (and takes value 0). Everywhere else this expression is fine; so the formally correct answer is

$$f^{-1}(x) = \begin{cases} \frac{-1+\sqrt{1+4x^2}}{2x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

4. Suppose f and g are increasing.

(a) Is $f + g$ necessarily increasing?

Solution: Yes. Suppose x, y are both in the domain of $f + g$ (so both x, y are both in the domains of both f and g separately) with $x < y$. We have $f(x) < f(y)$ and $g(x) < g(y)$ (by properties of f and g) so $(f + g)(x) < (f + g)(y)$.

(b) Is fg necessarily increasing?

Solution: Not necessarily; consider $f(x) = g(x) = x$ on $\mathbb{R}$.

(c) Is $f \circ g$ necessarily increasing?

Solution: Yes. Suppose x, y are both in the domain of $f \circ g$ (so both x, y are in the domain g, and both $f(x), f(y)$ are in the domain of f), with $x < y$. We have $g(x) < g(y)$ and so $(f \circ g)(x) = f(g(x)) < f(g(y)) = (f \circ g)(y)$.

5. On which intervals $[a, b]$ will the following functions by one-to-one?

(a) $f(x) = x^3 - 3x^2$.

Solution: f is continuous and differentiable on $\mathbb{R}$. $f'(x) = 3x^2 - 6x$, and this equals 0 at $x = 0, 2$. By examining test points in the intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$ we find that $f' > 0$ on the first, so f increasing, $f' < 0$ on the second, so f decreasing, and $f' > 0$ on the third, so f increasing.

Since f is continuous, we can add the end points to the various intervals to find that the *maximal* intervals on which f is monotone, and so one-to-one, are $A = (-\infty, 0]$, $B = [0, 2]$ and $C = [2, \infty)$. But of course, f is one-to-one on *any* subinterval of any of A, B, C, so (remembering that the question asked about finite *closed* intervals) the final answer is:

> The intervals $[a, b]$ on which f is one-to-one are those intervals $[a, b]$ that are completely contained in one of A, B, C above: $-\infty < a \leq b \leq 0$, or $0 \leq a \leq b \leq 1$ or $2 \leq a \leq b < \infty$.

(b) $f(x) = (1 + x^2)^{-1}$.

Solution: f is continuous and differentiable on $\mathbb{R}$. $f'(x) = -2x(1 + x^2)^{-2}$, and this equals 0 at $x = 0$. By examining test points in the intervals $(-\infty, 0)$ and $(0, \infty)$ we find that $f' > 0$ on the first, so f increasing, and $f' < 0$ on the second, so f decreasing.

Since f is continuous, we can add the end points to the intervals to find that the *maximal* intervals on which f is monotone, and so one-to-one, are $A = (-\infty, 0]$ and $B = [0, \infty)$. But of course, f is one-to-one on *any* subinterval of either of A, B, so (remembering that the question asked about finite *closed* intervals) the final answer is:

> The intervals $[a, b]$ on which f is one-to-one are those intervals $[a, b]$ that are completely contained in one of A, B above: $-\infty < a \leq b \leq 0$, or $0 \leq a \leq b < \infty$.

6. Find a formula for $(f^{-1})''(x)$, and decide under what circumstances the derivative actually exists.

Solution: For $f^{-1}(x)$ to be twice-differentiable, it is necessary for it to be differentiable. We know that this occurs iff $f'(f^{-1}(x))$ exists and is not 0, in which case

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Under what circumstances is this expression differentiable? We know from our study of the derivative in the fall that $1/g$ is differentiable at x iff g is differentiable at x, and $g(x) \neq 0$. Here, we know already that $f'(f^{-1}(x)) \neq 0$, so it is only necessary to further assume that $f''(f^{-1}(x))$ exists. Under that assumption, an application of chain rule and reciprocal rule yields

$$(f^{-1})''(x) = \left(\frac{1}{(f' \circ f^{-1})(x)}\right)' = \frac{-(f' \circ f^{-1})'(x)}{(f' \circ f^{-1})^2(x)} = \frac{-f''(f^{-1}(x))(f^{-1})'(x)}{\left(f'(f^{-1}(x))\right)^2} = \frac{-f''(f^{-1})(x)}{\left(f'(f^{-1}(x))\right)^3}.$$

The conditions that were *necessary* for this were

- $f'(f^{-1}(x))$ exists and is not 0 and
- $f''(f^{-1}(x))$ exists.

These are easily seen to be sufficient to allow the argument to run through.

7. Suppose that $f : [a, b] \to [c, d]$ is (strictly) increasing, and integrable on $[a, b]$. Prove that $f^{-1} : [c, d] \to [a, b]$ is integrable on $[c, d]$, and that in fact

$$\int_a^b f + \int_c^d f^{-1} = bd - ac.$$

Solution (sketch): Consider a partition $P = (t_0, \dots, t_n)$ (with $a = t_0 < t_1 < \cdots < t_n = b$) of $[a, b]$. Associated with this is a partition $P' = (f(t_0), \dots, f(t_n)$ (with $c = f(t_0) < f(t_1) < \cdots < f(t_n) = d$) of $[c, d]$.

From a picture, it is clear that the rectangles that make up the lower Darboux sum $L(f, P)$ together with the rectangles that make up the upper Darboux sum $U(f^{-1}, P')$, can be used to cover all of the square $[0, b] \times [0, d]$, except for an initial $[0, a] \times [0, b]$ square. In other words:

$$L(f, P) + U(f^{-1}, P') = bd - ac.$$

This can be made formal quite easily, using that f is increasing.

But also, the same argument shows

$$U(f, P) + L(f^{-1}, P') = bd - ac.$$

So we have

$$L(f,P) = bd - ac - U(f^{-1},P') \leq \int_a^b f \leq bd - ac - L(f^{-1},P') \leq U(f,P)$$

for every partition P of $[a,b]$. In particular, for any $\varepsilon > 0$, since there is a partition P of $[a,b]$ with $U(f,P) - L(f,P) < \varepsilon$, there is a partition P' of $[c,d]$ with

$$\big(bd - ac - L(f^{-1},P')\big) - \big(bd - ac - U(f^{-1},P')\big) < \varepsilon,$$

or

$$U(f^{-1},P') - L(f^{-1},P') < \varepsilon.$$

This shows that f^{-1} is integrable on $[c,d]$.

For the value of the integral, we have

$$bd - ac - U(f^{-1},P') \leq \int_a^b f \leq bd - ac - L(f^{-1},P')$$

or

$$L(f^{-1},P') \leq \left(\int_a^b f\right) - bd - ac \leq U(f^{-1},P'),$$

for all partitions of $[c,d]$ of the form P'. But in fact *all* partitions of $[c,d]$ can be expressed as P', for some partition P on $[a,b]$ (this easy fact uses invertibility of f, and the fact that f is increasing). This forces

$$\int_c^d f^{-1} = \left(\int_a^b f\right) - bd - ac,$$

as claimed.

8. Fix $a > 0$. Here is a scheme for defining a^x, for every rational x:

   Step 1 Set $a^0 = 1$ and set $a^n = a \cdot a^{n-1}$ for $n \in \mathbb{N}$ (we did this as an example of a recursive definition).

   Step 2 For $n \in \mathbb{N}$ define $a^{1/n}$ to be the unique positive x satisfying $x^n = a$ (we did this as an example of Intermediate Value Theorem).

   Step 3 For positive rational $r = m/n$ ($m, n \in \mathbb{N}$), set $a^r = (a^{1/m})^n$.

   Step 4 For negative rational r, set $a^r = 1/(a^{-r})$.

   The only questionable step is **Step 3**. A given rational has *many* representations of the form m/n, $m, n \in \mathbb{N}$; for example $2/3 = 8/12 = 100/150$.

   Check that the definition given in **Step 3** is in fact well-defined: if m/n and s/t are both representations of the same rational r, then **Step 3** gives the same value for a^r, whichever representation we use.

Solution: Let m, n, s and t be for natural numbers satisfying $m/n = s/t$. Our goal is to show that

$$(a^{1/n})^m = (a^{1/t})^s, \quad (\star)$$

where $a^{1/n}$ is that unique positive number such that $(a^{1/n})^n = 1$ and $a^{1/t}$ is that unique positive number such that $(a^{1/t})^t = 1$.

Since the function "raise x to the ntth power" is one-to-one on the positives (it is increasing in), we get that $(\star)$ is equivalent to

$$\left((a^{1/n})^m\right)^{nt} = \left((a^{1/t})^s\right)^{nt}. \quad (\star\star)$$

Now we need a general proposition about powers: for any real positive x, and any natural numbers y, z, we have

$$(x^y)^z = x^{yz}.$$

We prove this, for each fixed y, by induction on z. The base case $z = 1$ is clear (if states $(x^y)^1 = x^y$). For the induction step we have

$$\begin{aligned}
(x^y)^{z+1} &= (x^y)(x^y)^z \quad \text{(def. of } p^{q+1} \text{ for } q \in \mathbb{N}) \\
&= (x^y)(x^{yz}) \quad \text{(inductive hypothesis)} \\
&= x^{y+yz} \quad \text{(basic property proved in hwork last semester)} \\
&= x^{y(z+1)}.
\end{aligned}$$

Applying to both sides of $(\star\star)$ we find that $(\star\star)$ is equivalent to

$$(a^{1/n})^{mnt} = (a^{1/t})^{snt},$$

which, by a reverse application of the general proposition is equivalent to

$$\left((a^{1/n})^n\right)^{mt} = \left((a^{1/t})^t\right)^{sn}$$

which by definition is equivalent to

$$a^{mt} = a^{sn}.$$

But this is evidently true, since $m/n = s/t$ implies $mt = sn$.