Math 10860, Honors Calculus 2

Homework 4 **NAME**:

Due in class Frid♡y Febru♡ry 14

Instructions

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

> *I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!*

Justify your non-obvious assertions —

> *the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.*

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like "from this it follows that", or "it is now enough to prove ..., which we now do", etc), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I'll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Class notes Section 10.7 and Section 11 (my Section 11 is covered by Spivak's Chapter 12).

Assignment

1. Decide whether or not the following improper integrals exist.
   (a) $\int_0^\infty \frac{dx}{\sqrt{1+x^3}}$.
   (b) $\int_0^\infty \frac{dx}{x\sqrt{1+x}}$.
2. Suppose $\int_{-\infty}^\infty f$ exists. Let h, g be functions with $h(N) \to -\infty$ and $g(N) \to +\infty$ as $N \to +\infty$. Prove that
$$\lim_{N\to\infty} \int_{h(N)}^{g(N)} f$$

exists and equals $\int_{-\infty}^{\infty} f$.

3. Find f^{-1} for each of the following f. Specify the domain and range of f^{-1} in each case.
   (a) $f(x) = x^3 + 1$.
   (b) $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$
   (c) $f(x) = x + [x]$. (Remember that $[x]$ is the largest integer less than or equal to x.)
   (d) $f(x) = \frac{x}{1-x^2}$, $-1 < x < 1$.

4. Suppose f and g are increasing.
   (a) Is $f + g$ necessarily increasing?
   (b) Is fg necessarily increasing?
   (c) Is $f \circ g$ necessarily increasing?

5. On which intervals $[a, b]$ will the following functions by one-to-one?
   (a) $f(x) = x^3 - 3x^2$.
   (b) $f(x) = (1 + x^2)^{-1}$.

6. Find a formula for $(f^{-1})''(x)$, and decide under what circumstances the derivative actually exists.

7. In class we proved that if I is an interval, and that if f is continuous, invertible and has domain I, then f must be monotone. The proof used IVT, which in turn requires the completeness axiom.

   Show that the completeness axiom is necessary for any proof of this fact, by constucting, in "$\mathbb{Q}$-world", a ($\mathbb{Q}$-)function f defined on a ($\mathbb{Q}$-)interval, that is ($\mathbb{Q}$-)continuous and invertible, but for which f is not monotone.

8. Suppose that $f : [a, b] \to [c, d]$ is (stricktly) increasing, and integrable on $[a, b]$. Prove that $f^{-1} : [c, d] \to [a, b]$ is integrable on $[c, d]$, and that in fact

$$\int_a^b f + \int_c^d f^{-1} = bd - ac.$$

   Don't assume that f is continuous!

9. Fix $a > 0$. Here is a scheme for defining a^x, for every rational x:

   Step 1 Set $a^0 = 1$ and set $a^n = a \cdot a^{n-1}$ for $n \in \mathbb{N}$ (we did this as an example of a recursive definition).

   Step 2 For $n \in \mathbb{N}$ define $a^{1/n}$ to be the unique positive x satisfying $x^n = a$ (we did this as an example of Intermediate Value Theorem).

   Step 3 For positive rational $r = m/n$ ($m, n \in \mathbb{N}$), set $a^r = (a^{1/m})^n$.

Step 4 For negative rational r, set $a^r = 1/(a^{-r})$.

The only questionable step is **Step 3**. A given rational has *many* representations of the form m/n, $m, n \in \mathbb{N}$; for example $2/3 = 8/12 = 100/150$.

Check that the definition given in **Step 3** is in fact well-defined: if m/n and s/t are both representations of the same rational r, then **Step 3** gives the same value for a^r, whichever representation we use.

An extra credit problem: First, a few warm-up exercises whose solutions I won't look at, and aren't necessarily needed for the extra-credit problem, but might help motivate it and help you think about it:

- Suppose h is integrable on $[0, 1]$, that $h(x) \geq 0$ for all h, and that $\int_0^1 h = 0$. Must it be the case that $h(x) = 0$ for all x?
- Suppose h is *continuous* on $[0, 1]$, that $h(x) \geq 0$ for all h, and that $\int_0^1 h = 0$. Must it be the case that $h(x) = 0$ for all x?
- Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, and that for *every* continuous $g : [0, 1] \to \mathbb{R}$ we have $\int_0^1 fg = 0$. Prove that f is identically 0.

The last part above gives a way to "test" is a continuous function on a closed interval identically 0: hit it with an arbitrary continuous g, and if the integral of the product is always 0, then f is always 0. But in fact a smaller set of "test functions" exists. That's what the extra-credit problem is asking you to show:

> Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, and that for *every* continuous $g : [0, 1] \to \mathbb{R}$ *that satisfies* $g(0) = g(1) = 0$ we have
>
> $$\int_0^1 fg = 0.$$
>
> Prove that f is identically 0.