Math 10860, Honors Calculus 2

Homework 3 **NAME**:

Solutions

Instructions

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

> *I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!*

Justify your non-obvious assertions —

> *the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.*

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like "from this it follows that", or "it is now enough to prove ..., which we now do", etc), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I'll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Class notes Sections 11.4, 11.5 and 11.6, and/or Spivak Chapter 14, and Appendix to Chapter 8.

Assignment

1. Some questions on uniform continuity.

   (a) Recall that we argued in class that the function $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous. Find a function $f : (0,1] \to \mathbb{R}$ that is continuous but not uniformly continuous, *and is bounded on* $(0,1]$.

   Solution: What causes $f(x) = 1/x$ to be not uniformly continuous on $(0,1)$ is the fact that it has arbitrarily large slope close to 0. To get a bounded, non-uniformly continuous function on $(0,1)$, we want unbounded slope, but without the function being unbounded. One way to achieve that is to take $f(x) = \sin(1/x)$. This is certainly continuous on $(0,1)$. Is it *uniformly* continuous?[1] Take $\varepsilon = 1/10$, and consider $\delta > 0$. Is it possible that whenever $x, y \in (0,1)$ are within δ of each other, then $f(x), f(y)$ are within $1/10$ of each other? No! f takes on the values $+1$ and -1 inside every interval of the form $[1/(2\pi(n+1)), 1/(2\pi n)]$, and we can certainly find n large enough that the length of this interval is at most $\delta/2$; so we can find two points within $\delta/2$ of each other that have function values 2 apart, definitely more than $1/10$.

(b) Show that if $f, g : A \to \mathbb{R}$ are both uniformly continuous on A (some interval in $\mathbb{R}$), *and both bounded*, then fg is uniformly continuous on A.

Solution: Let $M > 0$ be a common bound for f, g on A; so $|f(x)|, |g(x)| \leq M$ for all $x \in A$. Note that

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leq M|g(x) - g(y)| + M|f(x) - f(y)|. \end{aligned}$$

Fix $\varepsilon > 0$. There is $\delta_1 > 0$ such that for $x, y \in A$, $|x-y| < \delta_1$ implies $|f(x)-f(y)| < \varepsilon/2M$, and $\delta_2 > 0$ such that for $x, y \in A$, $|x-y| < \delta_1$ implies $|g(x)-g(y)| < \varepsilon/2M$. Take $\delta = \min\{\delta_1, \delta_2\}$. We have that for $x, y \in A$, $|x-y| < \delta$ implies

$$|f(x)g(x) - f(y)g(y)| \leq M|g(x) - g(y)| + M|f(x) - f(y)| < \varepsilon.$$

This shows that fg is uniformly continuous on A.

(c) Give an example of an interval A, and functions $f, g : A \to \mathbb{R}$ that are both uniformly continuous on A, with f *not* bounded on A (but g is bounded on A), such that fg is not uniformly continuous on A.

Solution: A simple example is $f(x) = x$ and $g(x) = \sin x$ on $(0, \infty)$; both are easily seen to be uniformly continuous, and f is not bounded. The product $(fg)(x) = x \sin x$ is, however, *not* uniformly continuous on $(0, \infty)$: the derivative of the product is $x \cos x + \sin x$, which has intervals of fixed length where it is arbitrarily large.

2. Consider the function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

[1] Remember the definition: f is uniformly continuous on an interval I if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in I)((|x-y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon))$. Negating this, we find that the meaning of *not* uniformly continuous is that $(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x, y \in I)((|x-y| < \delta) \wedge (|f(x) - f(y)| \geq \varepsilon))$. In other words, to show that a function is *not* uniformly continuous on an interval, we just need to find a *single* $\varepsilon > 0$ that is bad, in the sense that every $\delta > 0$ has a witnessing x, y that are within δ of each other, but $f(x), f(y)$ are apart at least ε from each other.

Prove that there does not exist a function $g : [0, 2] \to \mathbb{R}$ with the property that $g' = f$.

Comment: It's tempting to think of *defining* the integral via

"$\int_a^b f = g(b) - g(a)$ where g is a function whose derivative is f."

One issue with this definition is that the function f may well be integrable on $[a, b]$, without there being any function g whose derivative is f. The function in this question (which certainly is integrable on $[0, 2]$) furnishes an example.

Solution: Because f is 0 on the interval $(0, 1)$ we know that g, if it exists, must be constant on $(0, 1)$, taking, say, value a, and similarly it must be constant on $(1, 2)$, taking, say, value a. Since g (if it exists) must be continuous at 1 (being differentiable there) it must be that $a = b = g(1)$, so that in fact g is constant on $(0, 2)$. But then $f(1) = g'(1) = 0$, a contradiction.

3. Find the derivatives of the following functions.

   (a) $F(x) = \int_a^{x^3} \sin^3 t \; dt$

   Solution: Define $g(x) = \int_a^x \sin^3 t \; dt$. By FTOC, $g'(x) = \sin^3 x$. We have that $F(x) = (g \circ c)(x)$ where $c(x) = x^3$, and so, by the chain rule,

$$\begin{aligned} F'(x) &= g'(c(x))c'(x) \\ &= \left(\sin^3 x^3\right) 3x^2. \end{aligned}$$

   (b) $F(x) = \int_x^{15} \left(\int_8^y \frac{dt}{1+t^2+\sin t} \right) dy$

   Solution: First some clean-up. We have

$$\int_x^{15} \left(\int_8^y \frac{dt}{1 + t^2 + \sin t} \right) dy = - \int_{15}^x \left(\int_8^y \frac{dt}{1 + t^2 + \sin t} \right) dy = - \int_{15}^x f(y) \; dy$$

   where

$$f(y) = \int_8^y \frac{dt}{1 + t^2 + \sin t}.$$

   By the FTOC, $F'(x) = -f(x) = - \int_8^x \frac{dt}{1+t^2+\sin t} = \int_x^8 \frac{dt}{1+t^2+\sin t}$.

   (c) $F(x) = \int_a^b \frac{x \; dt}{1+t^2+\sin^2 t}$

   Solution: We have

$$\int_a^b \frac{x \; dt}{1 + t^2 + \sin^2 t} = x \int_a^b \frac{dt}{1 + t^2 + \sin^2 t} = Cx$$

   where C is the constant $\int_a^b \frac{dt}{1+t^2+\sin^2 t}$. So

$$F'(x) = \int_a^b \frac{dt}{1 + t^2 + \sin^2 t}.$$

4. For each of the following functions f, consider $F(x) = \int_0^x f$, and determine at which points x is $F'(x) = f(x)$. Caution: there may be some x for which $F'(x) = f(x)$ even though the hypotheses of the obvious theorem do not apply.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$.

   Solution: Here
$$F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$$
   and so
$$F'(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$
   and $F'(1)$ does not exists. So:
$$F'(x) = f(x) \text{ for } x \neq 1.$$

   Note that f is continuous for all $x \neq 1$, and so we could have concluded (using the Fundamental Theorem of Calculus part 1) that $F' = f$ at all of these points, leaving us only to check what happens at $x = 1$; but this essentially requires figuring out what the function F looks like, so saves no work over the earlier approach. The same comment goes for the next two parts.

   (b) $f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$.

   Solution: Here
$$F(x) = 0 \quad \text{for all } x$$
   and so
$$F'(x) = 0 \quad \text{for all } x.$$
   So:
$$F'(x) = f(x) \text{ for } x \neq 1.$$

   (c) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$.

   Solution: Here
$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/2 & \text{if } x > 0 \end{cases}$$
   and so
$$F'(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0. \end{cases}$$
   So:
$$F'(x) = f(x) \text{ for all } x.$$

5. Let f be integrable on $[a, b]$, let c be in (a, b) and let

$$F(x) = \int_a^x f, \qquad a \le x \le b.$$

For each of the following statements, either give a proof or a counter-example.

(a) If f is differentiable at c then F is differentiable at c.

Solution: This is true. Since f is differentiable at c, it must be continuous at c, and so the FTOC part 1 gives that F is differentiable at c.

(b) If f is differentiable at c then F' is continuous at c.

Solution: Here it is very tempting to say "this is true", with the proof:

> If f is differentiable at c, then it is continuous at c, so F' exists (by FTOC) and equals f, which is continuous at c, so F' is continuous at c.

The problem is that we know only that $F' = f$ *at* c, not that $F' = f$ *in some neighborhood around* c; so we can't use the continuity of f at c to deduce the continuity of F' at c. Note that the above argument works to show that if f is continuous in some open interval that contains c, then F' is continuous on the interval, and so in particular at c.

If we only assume that f is continuous *at* c, then it is possible that arbitrarily close to c, F' does not exist, so that F' is not continuous at c. Spivak gives the following nice example: the function f which is

- 1 on $[1, \infty)$ and $(-\infty, -1]$
- $1/4$ on $[1/2, 1)$ and $(-1, -1/2]$
- $1/9$ on $[1/3, 1/2)$ and $(-1/2, -1/3]$
- $1/16$ on $[1/4, 1/3)$ and $(-1/3, -1/4]$
- and so on
- and is 0 at 0.

Because f is monotone increasing on $[0, \infty)$ it is integrable on $[0, a]$ for any $a > 0$, and because it is monotone decreasing on $(-\infty, 0]$ it is integrable on $[-a, 0]$ for any $a < 0$, so it is integrable on any interval containing 0, say $[-1, 1]$ for definiteness. It's easy to check that F' does not exist at ± 1, $\pm 1/2$, $\pm 1/3$, etc. (F is piecewise linear, and has sharp corners at each of these points), so it cannot be that F' exists at 0. But (exercise) f is differentiable at 0 with derivative 0.

(c) If f' is continuous at c, then F' is continuous at c.

Solution: This is true. The fact that f' is continuous at c means that f' is defined in some interval around c. On that interval, since f is differentiable, it is continuous. The (faulty) argument from the last part of this question can be used here to show that $F' = f$ on the interval on which f is continuous, and so F' in particular is continuous at c.

6. Two unrelated, but hopefully quick, parts.

(a) Show that *as x ranges over positive reals* the value of the following expression does not depend on x:

$$\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2},$$

and then (using this fact, or otherwise) deduce that

$$\int_0^1 \frac{dt}{1+t^2} = \int_1^\infty \frac{dt}{1+t^2}.$$

Solution: In fact, the expression *does* depend on x — my bad. Define a function f by $f(x) = \int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}$. The domain of this function is all reals (certainly f is defined for $x \neq 0$; it is just the sum of two finite integrals. For $x = 0$, the expression becomes $\int_0^\infty dt/(1+t^2)$, which we also know exists).

To show that the function doesn't depend on x, the obvious thing to do is to calculate $f'(x)$, using the Fundamental Theorem of Calculus. However, FTOC only applies to integrals of bounded functions defined on finite intervals, so FTOC will be able to say nothing about the function at $x = 0$. So we have to consider the function on the disjoint intervals $(-\infty, 0)$ and $(0, \infty)$.

Let us differentiate with respect to x. We have

$$\frac{d}{dx}\int_0^x \frac{dt}{1+t^2} = \frac{1}{1+x^2}$$

and

$$\frac{d}{dx}\int_0^{1/x} \frac{dt}{1+t^2} = \left(\frac{1}{1+(1/x)^2}\right)\frac{-1}{x^2} = \frac{-1}{1+x^2},$$

so that

$$\frac{d}{dx}\left(\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}\right) = 0.$$

So $f'(x) = 0$ for all $x \neq 0$. From this we conclude that the function is constant (i.e., does not depend on x) on the open interval $(0, \infty)$ (the interval of interest in the revised version of the question). It is also constant on the open interval $(-\infty, 0)$, where it actually takes a *different* constant value: for negative x,

$$\begin{aligned} f(x) &= \int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2} \\ &= -\int_x^0 \frac{dt}{1+t^2} - \int_{1/x}^0 \frac{dt}{1+t^2} \\ &= -\left(\int_0^{-x} \frac{dt}{1+t^2} - \int_0^{1/(-x)} \frac{dt}{1+t^2}\right) \\ &= -f(-x) \end{aligned}$$

using the fact, in the second-from-last line, that $1/(1+t^2)$ is an even expression. So f takes one constant value (a positive value) on $(0,\infty)$, and the negative of that value on $(-\infty,0)$.

Now for the second part: Let I be the value that $f(x)$ takes for all positive x. Taking $x=1$ we get:

$$I = \int_0^1 \frac{dt}{1+t^2} + \int_0^1 \frac{dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2}.$$

We know (we proved it in class) that $\int_0^\infty \frac{dt}{1+t^2} = \lim_{x\to\infty} \int_0^x \frac{dt}{1+t^2}$ exists. We claim that $\lim_{x\to\infty} \int_0^{1/x} \frac{dt}{1+t^2}$ also exists, and equals 0. Indeed, let $\varepsilon > 0$ be given. The maximum of $1/(1+t^2)$ on $[0,\infty)$ is 1, so for all $x>0$, $0 \le \int_0^{1/x} \frac{dt}{1+t^2} \le 1/x$. As long as $x > 1\varepsilon$ it follows that $0 \le \int_0^{1/x} \frac{dt}{1+t^2}\varepsilon$. This shows that $\lim_{x\to\infty} \int_0^{1/x} \frac{dt}{1+t^2} = 0$. It follows that

$$\lim_{x\to\infty}\left(\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}\right) = \int_0^\infty \frac{dt}{1+t^2}.$$

But also

$$\lim_{x\to\infty}\left(\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}\right) = \lim_{x\to\infty} I = I.$$

So we have

$$I = \int_0^\infty \frac{dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2}.$$

Subtracting $\int_0^1 \frac{dt}{1+t^2}$ from both sides, we get

$$\int_0^1 \frac{dt}{1+t^2} = \int_1^\infty \frac{dt}{1+t^2},$$

as claimed.

(b) Let f be a continuous function. Find $F'(x)$ if $F(x) = \int_0^x xf(t)\,dt$. **Hint**: the answer is *not* $xf(x)$.

Solution: Note that for each fixed x, we have $\int_0^x xf(t)\,dt = x\int_0^x f(t)\,dt$. Since f is continuous we can use the product rule for differentiation, and the first FTOC: from

$$F(x) = x\int_0^x f(t)$$

we get

$$F'(x) = xf(x) + \int_0^x f(t)\,dt.$$

An alternate approach *might* be: use the FTOC, part 2. For a fixed x, we have

$$F(x) = x\int_0^x f(t) = x(g(x)-g(0)) = xg(x) - xg(0)$$

where $g = g(t)$ is a function satisfying $g'(t) = f(t)$. Differentiating, we get

$$F'(x) = xg'(x) + g(x) - g(0) = xf(x) + \int_0^x f(t)\ dt.$$

The slight problem with this approach is that we have no certainty that such a g exists (c.f. a previous question on this homework).

7. Define $F(x) = \int_1^x \frac{dt}{t}$ and $G(x) = \int_b^{bx} \frac{dt}{t}$ (for $b \geq 1$).

   (a) Find $F'(x)$ and $G'(x)$.

   Solution: $F'(x) = 1/x$ (by FTOC) and $G'(x) = \frac{1}{bx} \cdot b = \frac{1}{x}$ (by FTOC and chain rule).

   (b) Use the result of the last part to answer the extra credit problem from Homework 1: for $a, b \geq 1$, prove that

   $$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

   Solution: Using basic properties, this is equivalent to

   $$\int_1^a \frac{dt}{t} = \int_b^{ab} \frac{dt}{t}.$$

   For fixed $b \geq 1$, considering both sides as functions of a we have that the derivatives of both sides are the same — this is exactly part a) of the problem. Hence there is an absolute constant C such that for all $a \geq 1$,

   $$\int_1^a \frac{dt}{t} = C + \int_b^{ab} \frac{dt}{t}.$$

   Evaluating at $a = 1$, we get $0 = \int_1^1 \frac{dt}{t} = C + \int_b^b \frac{dt}{t} = C + 0 = C$, so $C = 0$, and this completes the proof.

8. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\ dt$$

then

$$F'(x) = h(g(x))g'(x) - h(f(x))f'(x).$$

Solution: Let a be any constant. We have

$$\int_{f(x)}^{g(x)} h(t)\ dt = \int_{f(x)}^{a} h(t)\ dt + \int_a^{g(x)} h(t)\ dt = \int_a^{g(x)} h(t)\ dt - \int_a^{f(x)} h(t)\ dt.$$

(By continuity of h we know that all these integrals exist).

Now define $H(x) = \int_a^x h(t)\ dt$ (which exists for all x by continuity of h). We have that

$$\int_a^{g(x)} h(t)\ dt = (H \circ g)(x).$$

Since both H and g are differentiable (by the fundamental theorem of calculus and by hypothesis, respectively) we know that, by the chain rule for differentiation,

$$(H \circ g)'(x) = H'(g(x))g'(x) = h(g(x))g'(x)$$

(the last equality by the fundamental theorem of calculus again). Similarly

$$\int_a^{f(x)} h(t)\ dt = (H \circ f)(x)$$

and

$$(H \circ f)'(x) = H'(f(x))f'(x) = h(f(x))f'(x).$$

By linearity of the derivative,

$$F'(x) = h(g(x))g'(x) - h(f(x))f'(x).$$