Infinite limits/limits at infinity

Definitions:

- $\lim_{x\to a} f(x) = +\infty$ means: $(\forall M)(\exists\delta > 0)(\forall x)((0 < |x-a| < \delta) \Rightarrow (f(x) > M))$.
  (It's enough to prove this for all $M \geq M_0$, for any fixed M_0.)
- $\lim_{x\to a} f(x) = -\infty$ means: $(\forall M)(\exists\delta > 0)(\forall x)((0 < |x-a| < \delta) \Rightarrow (f(x) < M))$.
  (It's enough to prove this for all $M \leq M_0$, for any fixed M_0.)
- $\lim_{x\to a^+} f(x) = \infty$ means: $(\forall M)(\exists\delta > 0)(\forall x)((0 < x-a < \delta) \Rightarrow (f(x) > M))$.
  ($\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$ defined similarly.)
  All definitions above require f to be defined near a (or near a on one side, for one-sided limits); all definitions below require f to be defined "near $+\infty$" (or $-\infty$), meaning the domain of f includes all sufficiently large (or sufficiently large and negative) numbers.
- $\lim_{x\to\infty} f(x) = L$ means $(\forall\varepsilon > 0)(\exists M)(\forall x)((x > M) \Rightarrow (|f(x) - L| < \varepsilon))$.
- $\lim_{x\to-\infty} f(x) = L$ means $(\forall\varepsilon > 0)(\exists M)(\forall x)((x < M) \Rightarrow (|f(x) - L| < \varepsilon))$.
- $\lim_{x\to\infty} f(x) = \infty$ means $(\forall N)(\exists M)(\forall x)((x > M) \Rightarrow (f(x) > N))$.
  (Other variants — e.g. $\lim_{x\to-\infty} f(x) = -\infty$ — defined similarly.)

Theorems: All of these should make intuitive sense. You should treat at least some of them as exercises in working with formal ε-δ definitions.

- If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$, then
  - $\lim_{x\to\infty}(f+g)(x) = L + M$
  - $\lim_{x\to\infty}(fg)(x) = LM$
  - $\lim_{x\to\infty} cf(x) = cL$
  - $\lim_{x\to\infty}(f/g)(x) = L/M$, provided $M \neq 0$.
- At this point it is tempting to say that "there are lots of obvious variants of this list, too numerous to list, with L, M replaced variously by $\pm\infty$", but here's a **warning**:

  if $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$, then you can't draw any conclusion about $\lim_{x\to\infty}(f/g)(x)$ (think of the three cases 1) $f(x) = x^2$ and $g(x) = x$, 2) $f(x) = x$ and $g(x) = x^2$, 3) $f(x) = x$ and $g(x) = x$);

  if $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) = -\infty$, then you can't draw any conclusion about $\lim_{x\to\infty}(f+g)(x)$ (think of the three cases 1) $f(x) = x^2$ and $g(x) = -x^2 + 2$, 2) $f(x) = x^2 + x$ and $g(x) = -x^2$, 3) $f(x) = x^2 - x$ and $g(x) = -x^2$);

and

> if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = \infty$, then you can't draw any conclusion about $\lim_{x\to\infty}(fg)(x)$ (think of the three cases 1) $f(x) = 1/x$ and $g(x) = x$, 2) $f(x) = 1/x$ and $g(x) = x^2$, 3) $f(x) = 1/x^2$ and $g(x) = x$).

(We'll address situations like this more carefully when we talk about L'Hospital's rule.)

- Some things can be said: e.g., if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = +\infty$, then
  - $\lim_{x\to\infty}(f+g)(x) = +\infty$
  - $\lim_{x\to\infty}(fg)(x) = \begin{cases} +\infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0, \end{cases}$

  and if $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$ then
  - $\lim_{x\to\infty}(f+g)(x) = +\infty$
  - $\lim_{x\to\infty}(fg)(x) = +\infty$.
- If $L \neq 0$,
  - $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0^+} f(1/x) = 1/L$
  - $\lim_{x\to-\infty} f(x) = L$ if and only if $\lim_{x\to 0^-} f(1/x) = 1/L$.

  (So limits at infinity can be converted to ordinary limits.)
- For $n \in \mathbb{N} \cup \{0\}$,
  - $\lim_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \end{cases}$ and
  - $\lim_{x\to\infty} \frac{1}{x^n} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$
- Suppose $p(x)$ is the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, and $q(x)$ is the polynomial $q(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$ $(n, m \geq 0)$. Then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 1 & \text{if } n = m \\ \infty & \text{if } n > m \\ 0 & \text{if } n < m. \end{cases}$$

- This list of results is far from exhaustive. Any other property of infinite limits/limits at infinity that you come up with, that seems reasonable, almost certainly is true; if you have any doubt you should just go ahead and prove the property from the definitions. Just be careful when adding infinities of opposite sign, and when taking the ratios of infinities (regardless of sign). Also be careful when multiplying 0 by any infinity. Multiplying infinities (of any sign) is never a problem.