Math 10850, Honors Calculus 1

Quiz 9, Thursday December 5

Solutions

1. Give a clear and complete statement of the Mean Value Theorem.

   Solution: If f is continuous on $[a,b]$ and differentiable on (a,b) then there is a number $c \in (a,b)$ with

   $$\frac{f(b)-f(a)}{b-a} = f'(c).$$

2. This part concerns the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^3 + 3x^2 + 6x - 12$. Say that a real number c is a *real root* of f if $f(c) = 0$.

   (a) Show that f does not have two (or more) real roots.

   Solution: Assume (for a contradiction) that there are real roots $a < b$. On the closed interval $[a,b]$, f is continuous, and it's differentiable on (a,b), so by the mean value theorem there is some number $c \in (a,b)$ with $f'(c) = 0$.

   But $f'(x) = 6x^2 + 6x + 6x = 6(x^2 + x + 1)$, and this is positive for all real x ($x^2 + x + 1 = 0$ only when $x = (-1 \pm \sqrt{-3})/2$, neither of which are real numbers). So there is not a real c with $f'(c) = 0$.

   This contradiction shows that f cannot have two real roots (or more).

   One could also argue as follows: since $f'(x) = 6x^2 + 6x + 6x = 6(x^2 + x + 1)$, and $6(x^2 + x + 1) > 0$ for all real x, we have that f is strictly increasing on its whole domain. So if c is a real root, then for $d > c$ we have $f(d) > f(c)$ and so $f(d) \neq 0$, and for $c > e$ we have $f(c) > f(e)$ and so $f(e) \neq 0$. So f can have at most one real root.

   (b) Show that f has *exactly* one real root.

   Solution: We have shown that f can have at most one real root, so it remains to show that it has at least one real root.

   Note that $f(0) = -12 < 0$ and $f(2) = 28$. f is continuous on the closed interval $[0,2]$, and goes from negative to positive on the interval; so by the Intermediate Value Theorem, there is $c \in (0,2)$ with $f(c) = 0$. So f indeed has at least one real root.