Math 10850, Honors Calculus 1

Quiz 6, Thursday October 17

Solutions

1. Give a precise (ε-δ) definition of what it means to say that a function f approaches a limit L near a.

   Solution: Prose answer:

   > A function f approaches a limit L near a if
   > - f is defined near a,
   >
   > and if
   > - for all positive $\varepsilon > 0$ there is a positive δ such that whenever x is within distance δ of a (but not equal to a), $f(x)$ is within distance ε of L.

   Symbolic answer:

   > $\lim_{x \to a} f(x) = L$ if
   > - $(\exists \Delta > 0)((a - \Delta, a) \cup (a, a + \Delta) \subseteq \text{Domain}(f))$
   >
   > and
   > - $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((0 < |x - a| < \delta) \Rightarrow |f(x) - L| < \varepsilon)$.

   Of course, a mix of prose and symbols is fine.

   Note 1: it is important, when presenting the definition symbolically, to quantify over all x. Without a quantification over x, the expression becomes a predicate (depending on x), so may be true or false depending on the particular choice of x; with the quantification over x, the expression becomes a *statement* (maybe true, maybe false, but with a definitive truth value.)

   Note 2: the condition "f defined near a" is somewhat important, in that it helps clarify the definition, but is not critical. Indeed, suppose that f is *not* defined near a. This means that for *every* $\delta > 0$, there is *some* $x \in (a - \delta, a + \delta)$ for which $f(x)$ is not defined. This says that f *does not* tens to *any* limit near a, because no matter how small a δ is chosen, there will be at least one x that satisfies the premise $(0 < |x - a| < \delta)$ of the implication, but fails to satisfy the conclusion $(|f(x) - L| < \varepsilon)$, by virtue of $f(x)$ not existing. So, bottom line, you can get away without the condition "f defined near a".

2. Prove (directly using the definition of limit) that $\lim_{x \to 2} \frac{x-1}{x+1} = \frac{1}{3}$.

   Solution: Certainly f is defined near 2. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$ then

   $$\left| \frac{x-1}{x+1} - \frac{1}{3} \right| < \varepsilon.$$

   Now

   $$\begin{aligned} \left| \frac{x-1}{x+1} - \frac{1}{3} \right| &= \left| \frac{3(x-1) - (x+1)}{3(x+1)} - \frac{1}{3} \right| \\ &= \frac{2|x-2|}{3|x+1|}. \end{aligned}$$

   If $\delta \leq 1$ then $0 < |x - 2| < \delta$ implies $|x - 2| < 1$, which implies $x \in (1, 3)$, which implies $x + 1 \in (2, 4)$, which implies

   $$|x + 1| > 2 \quad \text{and so} \quad \frac{2}{3|x+1|} < \frac{1}{3}.$$

If *also* $\delta \leq 3\varepsilon$ then $0 < |x-2| < \delta$ implies $|x-2| < 3\varepsilon$, and

$$\frac{2|x-2|}{3|x+1|} < (3\varepsilon)\left(\frac{1}{3}\right) = \varepsilon.$$

So if we choose $\delta = \min\{1, 3\varepsilon\}$, have that $0 < |x-2| < \delta$ implies

$$\left|\frac{x-1}{x+1} - \frac{1}{3}\right| = \frac{2|x-2|}{3|x+1|} < \varepsilon.$$

This shows that $\lim_{x\to 2} \frac{x-1}{x+1} = \frac{1}{3}$.

Note: The choice $\delta \leq 1$ initially was arbitrary, just to ensure that $x+1$ stayed away from 0. "1" could have been replaced with anything smaller than 3; a different choice would lead to a different constant dividing/multiplying ε. There is no single "right" choice.