Math 10850, fall 2019, University of Notre Dame

Notes on first exam

September 29, 2019

The key facts

The first midterm will be on

Friday, October 4, in class.

The class runs 50 minutes, but since the next class does not meet until 30 minutes after ours, the exam will run for **1 hour**.

What you need to know

- Logic:
  - What is a statement (a.k.a. proposition)?
  - What are conjunction, disjunction, negation, implication, bidirectional implication (if and only if)?
  - What are the contrapositive and converse of an implication?
  - What is a truth table?
  - What are the truth tables of conjunction, disjunction, etc.?
  - What is a tautology?
  - How do you use a truth table to check if a proposition is a tautology or a contradiction?
  - What are the basic logical equivalences such as associativity, commutativity, distributivity, De Morgan's laws, the law of implication?
  - What is a predicate?
  - What are universal and existential quantification?
  - How do you negate a quantified predicate?
- Axioms of real numbers

- What are the four axioms that characterize the basic properties of addition?
- What about the axioms of multiplication?
- What is the distributive axiom?
- What are the three axioms of order?
- How are the basic symbols of inequality defined?
- What is an upper bound, and a least upper bound, for a set?
- What is the completeness axiom?
- How do you derive other basic properties of numbers from the axioms?
- What is the absolute value function?
- What is the triangle inequality?
- How do you solve equalities and inequalities involving the absolute value function?
- What is an inductive set? (You should know this, but I won't examine it.)
- How are the natural numbers defined?
- What is the principle of mathematical induction?
- What is the principle of complete induction?
- How do you prove that addition of many terms is associative?
- What is the well-ordering principle?
- How do you structure a proof by induction?
- What is summation notation?
- What is a definition of a sequence of terms by recursion?
- What is the binomial coefficient $\binom{n}{k}$?
- What is the binomial theorem (concerning $(x+y)^n$)?
- What is the Bernoulli inequality?
- What are the integers, and the rationals?
- How do you show that $\sqrt{2}$ is irrational?

• Functions

- What is a function (of real numbers), both informally and formally?
- What are domain, range and codomain?
- What are the sum, difference, product and ratio/quotient (division) of functions?
- How is the domain of a function built from additional, multiplication, etc., obtained from the domain of the constituent parts?

  - What are constant, linear, power, polynomial and rational functions?
  - What is the composition of functions?
  - How is the domain of a function built from composition obtained from the domain of the constituent parts?
  - What is the graph of a function?
  - What do the graphs of constant, linear, power and polynomial functions look like?
  - What are circles and ellipses?
  - What are the sin and cosine functions?

You should be able to prove things like:

- $(-a)(-b) = ab$
- $ab = 0$ implies one of a, b is 0
- If $x, y \geq 0$, then $x^2 \geq y^2$ is equivalent to $x \geq y$
- the triangle inequality
- negative times positive is negative, negative times negative is positive
- the general associative property
- generalizations to n terms of many of the properties that we have established for two terms — commutativity, triangle inequality, distributivity, et cetera
- the binomial theorem
- Bernoulli's inequality
- the irrationality of $\sqrt{2}$

and you should be able to give formal, correct and complete statements of definitions, concepts and axioms such as

- commutativity, associativity, and existence of identity and inverses for addition and multiplication
- the distributive property of numbers
- the trichotomy axiom
- the closure axioms for positivity
- greater than, less than

- upper bounds and least upper bounds
- the completeness axiom
- inductive sets and natural numbers (again, you *should* know this, but I won't ask about it)
- the principle of induction, the principle of strong induction, the well-ordering principle
- functions, domain, codomain, range
- composition

Practice questions

These are not intended to be "clones" of the actual exam questions; rather, they are intended to be questions that get you thinking about all the material that be examined.

Although many of the questions don't explicitly say this, it is a given that you **justify** all statements you make! So, for example, in the last part of question 2, you would get no credit for answering "True, True, False" (if these happened to be the right answers), without explaining your reasoning in each case (a short proof that the true assertions are true, and an example to show that the false one is false).

These questions were written quickly and may have some lack of clarity. I will try very hard to make sure that what I expect from you from an exam question is crystal clear. But to help in this, please tell over the coming days if you find instructions in these questions to be unclear or confusing.

1. (a) If p and q are logical starements, say what the meaning of $p \Rightarrow q$ is. (I'm not looking for an essay or a discussion; just a short, precise definition of the symbol "$\Rightarrow$".)

   (b) Which of the following are equivalent to $\neg(p \Rightarrow r) \Rightarrow \neg q$? There may be more than one, or none. (In the list below, I'm using "$\rightarrow$" for "$\Rightarrow$")

   (1) $\neg(p \rightarrow r) \vee q$
   (2) $(p \wedge \neg r) \vee q$
   (3) $(\neg p \rightarrow \neg r) \vee q$
   (4) $q \rightarrow (p \rightarrow r)$
   (5) $\neg q \rightarrow (\neg p \rightarrow \neg r)$
   (6) $\neg q \rightarrow (\neg p \vee r)$
   (7) $\neg q \rightarrow \neg(p \rightarrow r)$

   (c) Show that $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology

   i. via a truth table, and
   ii. via a sequence of logically equivalent propositions, starting with $(p \wedge q) \Rightarrow (p \vee q)$ and ending with T.

2. Here are three mathematical statements:

   - For each $m, n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $m < p$ and $p < n$.
   - For all non-negative real numbers a, b and c, if $a^2 + b^2 = c^2$ then $a + b \geq c$.
   - There does not exist a positive real number a such that $a + (1/a) < 2$.

   For each of the three,

   (a) express it using predicates and quantifiers. In each case say what the universe of discourse is for each variable, and define all your predicates. For the second statement above, you **must** take the universe of discourse for a, b and c to be all reals.

   (b) Then negate each statement, simplifying as much as you can (in particular, the negations should not contain any implications).

   (c) Which of the three statements is true, and which is false?

3. The exclusive-OR operator $\vee^\star$ has the following truth table:

p	q	$p \vee^\star q$
T	T	F
T	F	T
F	T	T
F	F	F

   (a) Show that $p \vee^\star q$ is logically equivalent to $(p \Rightarrow \neg q) \wedge (\neg p \Rightarrow q)$.

   (b) Write down an expression involving p, q and one or both of the logical operators $\wedge$ and $\neg$, that is logically equivalent to $p \vee^\star q$.

4. (a) State all of the axioms of real numbers that mention addition (there are six of them).

   (b) Give a careful proof, only using the axioms, with *every* step justified, that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

   as long as $b, d \neq 0$.

   (c) Give a careful proof, only using the axioms, with *every* step justified, that

$$(ab)^{-1} = a^{-1}b^{-1}$$

   as long as $a, b \neq 0$.

(d) Find, with proof, all real numbers x such that $x^4 = x$. (Here you only need to justify any non-obvious step; for example, you don't need to tell me that you are using associativity, but you do need to tell me if you are using that $(-a)(-b) = ab$.)

5. (a) Find all x satisfying the inequality $-2 < |x-1| - |x+2| \leq 2$.

   (b) Find all real x such that $x^2 - x + 10 > 16$.

   (c) Express $|(|x| - |x^2|)|$ without absolute value signs (use brace notation if necessary, to treat cases).

   (d) Verify that for all real x, y, $|xy| = |x||y|$.

6. (a) State the binomial theorem (concerning the expansion of $(x+y)^n$), and explain how to calculate the expression $\binom{n}{k}$ that appears in the theorem.

   (b) What is the coefficient of a^6b^8 in $(2a - b^2/2)^{10}$?

   (c) Prove that for all $m, n \geq 0$,

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n}.$$

   Also, draw Pascal's triangle, and circle all the entries of the triangle that are involved in this identity, for some values of m and n. The identity is sometimes called the *hockey stick* identity — does this name make sense?

7. (a) Give a proof, from the axioms, that if $a > b$ then $-a < -b$. **Note**: You should remember that none of the axioms mention the symbols $<$ and $>$, so the first thing you need to do is to translate this into a statement about positivity. You should attempt to prove this result *only* using the axioms, which may mean that you will need to prove a few auxiliary statements along the way. In the end, the proof will probably be quite long, and a good review of the process of deriving basic properties of the real numbers from the axioms.

   (b) We all know that $(a+b)(c+d) = ac + ad + bc + bd$. Using *only* the axioms of real numbers, verify that the expression on the right is well-defined (i.e., has only one possible value), and verify that the equation is true. How many axioms do you need to appeal to?

8. Find the flaw in the following proof that all natural numbers equal 1. (Taken from `https://faculty.math.illinois.edu/~hildebr/347.summer14/induction4.pdf`)

Claim: $\boxed{\text{All positive integers are equal}}$

Proof: To prove the claim, we will prove by induction that, for all $n \in \mathbb{N}$, the following statement holds:

$(P(n))$ For any $x, y \in \mathbb{N}$, if $\max(x, y) = n$, then $x = y$.

(Here $\max(x, y)$ denotes the larger of the two numbers x and y, or the common value if both are equal.)

Base step: When $n = 1$, the condition in $P(1)$ becomes $\max(x, y) = 1$. But this forces $x = 1$ and $y = 1$, and hence $x = y$.

Induction step: Let $k \in \mathbb{N}$ be given and suppose $P(k)$ is true. We seek to show that $P(k + 1)$ is true as well.

Let $x, y \in \mathbb{N}$ such that $\max(x, y) = k+1$. Then $\max(x-1, y-1) = \max(x, y)-1 = (k + 1) - 1 = k$. By the induction hypothesis, it follows that $x - 1 = y - 1$, and therefore $x = y$. This proves $P(k + 1)$, so the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. In particular, since $\max(1, n) = n$ for any positive integer n, it follows that $1 = n$ for any positive integer n. Thus, all positive integers must be equal to 1

9. (a) Suppose that $g = h \circ f$. Prove that if $f(x) = f(y)$ then $g(x) = g(y)$.

   (b) Suppose that f and g are two functions such that $g(x) = g(y)$ whenever $f(x) = f(y)$. Construct a function h such that $g = h \circ f$. [Try to define $h(z)$ when z is of the form $z = f(x)$ for some x (these are the only z that matter), then use the hypothesis to show that your definition does not run into trouble.]

Questions on functions and graphs

Many of these will appear on the homework due Friday, October 11.

1. Let $f(x) = 1/(1 + x)$.

   (a) What is $f(f(x))$? And what is the domain of this new function?

   (b) What is $f(cx)$, where c is some fixed real number? And what is the domain of this new function?

   (c) For which real numbers c, is there a number x such that $f(cx) = f(x)$?

   (d) For which numbers c is it true that $f(cx) = f(x)$ for (at least) two different numbers x?

2. Find the domain of each of these functions.

   (a) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$.

   (b) $f(x) = 1/(x - 1) + 1/(x - 2)$.

   (c) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$.

3. A function f is said to be *even* if $f(x) = f(-x)$ for all x, and *odd* if $f(x) = -f(-x)$ for all x (so $f(x) = |x|$ is even and $f(x) = x^3$ is odd, for example).

   (a) Determine whether $f + g$ is even, odd, or not necessarily either in the four cases obtained by choosing f even or odd, and g even or odd.

   (b) Do the same for $f \cdot g$.

   (c) Do the same for $f \circ g$.

   (d) Prove that every even function f can be written as $f(x) = g(|x|)$, for *infinitely many* functions g.

4. For each of the following assertions, either give a proof (if the assertion is true) or a counterexample (if it is false).

   (a) $f \circ (g + h) = f \circ g + f \circ h$.

   (b) $(g + h) \circ f = g \circ f + h \circ f$.

   (c) $1/(f \circ g) = (1/f) \circ g$.

   (d) $1/(f \circ g) = f \circ (1/g)$.

5. Indicate on the real number line the set of all x satisfying the following conditions. Also express each set in interval notation (possibly using $\cup$).

   (a) $|x^2 - 1| < 1/2$.

   (b) $1/(1 + x^2) \leq a$ (the answer may depend on a, so you may have to consider cases).

6. Sketch in the coordinate plane the set of points (x, y) satisfying:

   (a) the inequality $x > y$.

   (b) the inequality $|x - y| < 1$.

   (c) the condition $x + y \in \mathbb{Z}$.

   (d) $|x| + |y| = 1$.

   (e) $y^2 > 2x^2$.

7. The symbol $[x]$ denotes the largest integer which is $\leq x$; it's called the *integer part* of x. So, for example, $[2.1] = 2$, $[2] = 2$, $[-0.9] = -1$ and $[-1] = -1$.. Draw the graph of the following functions:

   (a) $f(x) = [x]$.

   (b) $f(x) = [1/x]$.

8. (a) Give the formal definition of a function, and give the formal definition of the *domain* of a function.

(b) Find the domains of each of the following functions:

i. $f(x) = \sqrt{1-x} + \sqrt{2-x}$

ii. $g(x) = 1/\sqrt{x^2 - 5x + 6}$

iii. $h_2 = h_1 \circ h_1$ where $h_1 = -1/x$ for $x > 0$ and undefined otherwise.

9. (a) Suppose that $g = h \circ f$. Prove that if $f(x) = f(y)$ then $g(x) = g(y)$.

(b) Suppose that f and g are two functions such that $g(x) = g(y)$ whenever $f(x) = f(y)$. Construct a function h such that $g = h \circ f$. [Try to define $h(z)$ when z is of the form $z = f(x)$ for some x (these are the only z that matter), then use the hypothesis to show that your definition does not run into trouble.]