Math 10850, Honors Calculus 1

Homework 5

Due in class Friday October 11

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

Section 5 of the course notes (functions & graphs), and/or Spivak Chapter 3 & 4.

Assignment

1. Let $f(x) = 1/(1+x)$.

   (a) What is $f(f(x))$? And what is the domain of this new function?

   (b) What is $f(cx)$, where c is some fixed real number? And what is the domain of this new function?

   (c) For which real numbers c, is there a number x such that $f(cx) = f(x)$?

   (d) For which numbers c is it true that $f(cx) = f(x)$ for (at least) two different numbers x?

2. Find the domain of each of these functions.

   (a) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$.

   (b) $f(x) = 1/(x-1) + 1/(x-2)$.

   (c) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$.

3. A function f is said to be *even* if $f(x) = f(-x)$ for all x, and *odd* if $f(x) = -f(-x)$ for all x (so $f(x) = |x|$ is even and $f(x) = x^3$ is odd, for example).

   (a) Determine whether $f + g$ is even, odd, or not necessarily either in the four cases obtained by choosing f even or odd, and g even or odd. (**Note**: here, if you think that $f + g$ is (say) even, then you should show why $(f + g)(-x) = (f + g)(x)$ follows from whatever properties of f, g you are assuming. If you think that it's not possible to say definitely whether $f + g$ is odd or even, you should give explicit examples. Typically, it will suffice to produce a *single* example of a pair f, g with $f + g$ *neither* even *nor* odd — witnesses by an x with $(f+g)(-x) \neq (f+g)(x)$ and $(f+g)(-x) \neq -(f+g)(x)$.)

(b) Do the same for $f \cdot g$.

(c) Do the same for $f \circ g$.

(d) Prove that every even function f can be written as $f(x) = g(|x|)$, for *infinitely many* functions g.

4. For each of the following assertions, either give a proof (if the assertion is true) or a counterexample (if it is false).

   (a) $f \circ (g + h) = f \circ g + f \circ h$.

   (b) $(g + h) \circ f = g \circ f + h \circ f$.

   (c) $1/(f \circ g) = (1/f) \circ g$.

   (d) $1/(f \circ g) = f \circ (1/g)$.

5. Indicate on the real number line the set of all x satisfying the following conditions. Also express each set in interval notation (possibly using $\cup$).

   (a) $|x^2 - 1| < 1/2$.

   (b) $1/(1 + x^2) \leq a$ (the answer may depend on a, so you may have to consider cases).

In the following two questions, I'm asking you to draw graphs/regions of the plane. **Please** draw your axes using a straightedge; label both axes, and include a scale on both axes! (Otherwise, your graph is not fully interpretable). Give your graphs a little space — don't try to scrunch two of them side-by-side on the page. When an endpoint of a continuous[1] piece of your graph is not part of the graph, indicate this by a hollow circle. When it is part of the graph, indicate this by a solid circle.

6. Sketch in the coordinate plane the set of points (x, y) satisfying:

   (a) the inequality $x > y$.

   (b) the inequality $|x - y| < 1$.

   (c) the condition $x + y \in \mathbb{Z}$.

   (d) $|1 - x| = |y - 1|$

   (e) $x^2 - 2x + y^2 = 4$.

   (f) $y^2 > 2x^2$.

7. The symbol $[x]$ denotes the largest integer which is $\leq x$; it's called the *integer part* of x. So, for example, $[2.1] = 2$, $[2] = 2$, $[-0.9] = -1$ and $[-1] = -1$.. Draw the graph of the following functions:

[1] Continuous? What does that word mean? I have no idea. But, I'll learn about it in the next week or so, and tell you in class...

(a) $f(x) = [x]$.

(b) $f(x) = [1/x]$.

(c) $f(x) = \sqrt{x - [x]}$.

8. Find the domains of each of the following functions:

   (a) $f(x) = \sqrt{1-x} + \sqrt{2-x}$

   (b) $g(x) = 1/\sqrt{x^2 - 5x + 6}$

   (c) $h_2 = h_1 \circ h_1$ where $h_1 = -1/x$ for $x > 0$ and undefined otherwise.

9. A *parabola* is the set of points in the plane with the following characteristic property: it is the set of points (x, y) such that the distance from (x, y) to a fixed point (a, b) is equal to the distance[2] from (x, y) to a fixed line L (that does not pass through (a, b)). Succinctly, a parabola is the set of points equidistant from a fixed point and a fixed line.

   A parabola is not always the graph of some function, but when it is, it is the graph of a quadratic function; and conversely, the graph of a quadratic function is a parabola. This question (essentially) asks you to prove this.

   (a) Let L be the horizontal line $y = c$ (c some real constant) and let P be the point (a, b), with $b \neq c$. Prove that the parabola determined by L and P is a set of points of the form $(x, rx^2 + sx + t)$ for some real constants r, s, t (i.e., the parabola is the graph of a specific quadratic function).

   (b) Let $f(x) = ax^2 + bx + c$ be a quadratic function, with $a > 0$. Find a horizontal line L and a point P, not on L, such that the parabola determined by L and P is exactly the graph of f.

An extra credit problem

Please submit this on a *separate* sheet.

The Dirichlet function, defined by

$$\text{Dir}(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

has the property that on *every* open interval (a, b) (however small) it takes on both values 0 and 1.

Find a more remarkable function. Specifically, find a function $f : \mathbb{R} \to \mathbb{R}$ that has the property that on every open interval (a, b) (however small), f takes on *every* rational value.

[2]The distance from a point P to a line L is defined to be the distance from P to that point on L that is closest to P. It is the distance from P to L along that line through P that is perpendicular to P. If L the horizontal line $y = c$ (c some real constant) and P is the point (a, b), then the distance from P to L is easy to compute: it is $|b - c|$.