# Math 10850, Honors Calculus 1

## Homework 1

## Solutions

1. Carefully and completely read the sections of this document headed "General notes on the homework" and "Writing homework solutions", and ask me if you need any clarification!

2. Do the assigned reading for this weeks' homework.

3. (a) Verify, using a truth table, that the proposition "not ($p$ implies $q$)" is equivalent to the proposition "$p$ and (not $q$)".

   **Solution**: Here is a single truth-table that deals with "not ($p$ implies $q$)" and "$p$ and (not $q$)" simultaneously:

| $p$ | $q$ | $p \Rightarrow q$ | not ($p$ implies $q$) | $\neg q$ | $p$ and (not $q$) |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ |

   Comparing the 4th and 6th columns we see that they are identical, so indeed $\neg(p \Rightarrow q)$ is equivalent to $p \wedge (\neg q)$. Notice that I'm engaging in some over-kill here; it is perfectly reasonable to go straight from the first two columns to the columns for $\neg(p \Rightarrow q)$ and $p \wedge (\neg q)$. In the future I probably won't use intermediate columns so much.

   (b) Show that the negation of the proposition $p \Rightarrow q$ (so, the proposition "It is not the case that $p$ implies $q$") is *not* equivalent to any implication that involves

   - one of $p$, $\neg p$, $q$ or $\neg q$ on one side
   - one of $p$, $\neg p$, $q$ or $\neg q$ on the other side
   - exactly one occurrence of one of $p$, $\neg p$ in total, and exactly one occurrence of one of $q$, $\neg q$ in total.

(There is a slow way to do part (b), by checking all possible cases. That seems like a pain! Can you find a fast way?)

**Solution**: The slow way is to check the truth tables of each of

- $p \Rightarrow q$,
- $p \Rightarrow (\neg q)$,
- $(\neg p) \Rightarrow q$,
- $(\neg p) \Rightarrow (\neg q)$,
- $q \Rightarrow p$,
- $q \Rightarrow (\neg p)$,
- $(\neg q) \Rightarrow p$,
- $(\neg q) \Rightarrow (\neg p)$,

and verify that none of them agrees with $\neg(p \Rightarrow q)$.

The fast way: each of the 8 implications we suggested checking above are *implications*, with the premise being completely independent of the hypothesis (one only involving $p$, one only involving $q$) and so will have exactly one $F$ in the last column of their truth tables. But $\neg(p \Rightarrow q)$ is the *negation* of an implication, so will have exactly one $T$ in the last column of its truth table, so exactly three $F$'s. Since $1 \neq 3$, none of the eight implications can have the same truth table as $\neg(p \Rightarrow q)$.

4. (This question provides another justification for the definition/truth table of $p \Rightarrow q$.)

Suppose we hadn't defined implication, and are trying to "reason out" a truth table for $p \Rightarrow q$. We should all agree that if $p$ is true and $q$ is true then $p \Rightarrow q$ is true (is a valid contract/promise), and that if $p$ is true but $q$ is false then $p \Rightarrow q$ is false (is an invalid contract/promise). The rest of the truth table for $p \Rightarrow q$ is more problematic :(.

There are four possible ways of completing the truth table, leading to four "candidate" truth tables for $p \Rightarrow q$. Here are the four, which I have chosen to label $p \Rightarrow_1 q$, $p \Rightarrow_2 q$, $p \Rightarrow_3 q$ and $p \Rightarrow_4 q$:

| $p$ | $q$ | $p \Rightarrow_1 q$ | $p \Rightarrow_2 q$ | $p \Rightarrow_3 q$ | $p \Rightarrow_4 q$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ |

Notice that $p \Rightarrow_1 q$ is the one that agrees with $p \Rightarrow q$, as we have defined it.

Another aspect of implication that we should all agree on is the fundamental rule of inference, formally referred to as *modus ponens*, which says that if we known that $p$ is true and also that $p$ implies $q$, then we can infer that $q$ is true. In other words, it should be the case that

$$(p \wedge (p \Rightarrow q)) \Rightarrow q$$

is a tautology (an absolutely true statement — T's in every row of the final column of the truth table).

(a) Which of the following are tautologies? (Construct truth tables for each.)

i. $(p \wedge (p \Rightarrow_1 q)) \Rightarrow_1 q$

ii. $(p \wedge (p \Rightarrow_2 q)) \Rightarrow_2 q$

iii. $(p \wedge (p \Rightarrow_3 q)) \Rightarrow_3 q$

iv. $(p \wedge (p \Rightarrow_4 q)) \Rightarrow_4 q$

**Solution**: Here is a combined truth table of each of $(p \wedge (p \Rightarrow_i q)) \Rightarrow_i q$, $i = 1, 2, 3, 4$:

| $p$ | $q$ | $p \Rightarrow_i q$ | $p \wedge (p \Rightarrow_i q)$ | $q$ | $(p \wedge (p \Rightarrow_i q)) \Rightarrow_i q$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $Y$ |
| $F$ | $T$ | $X$ | $F$ | $T$ | $X$ |
| $F$ | $F$ | $Y$ | $F$ | $F$ | $Y$ |

The values of $X$ and $Y$ depend on which $i$ is under consideration; for example, when $i = 2$ we have $X = T$ and $Y = F$.

Notice that because both $X$ and $Y$ appear in the final column, $(p \wedge (p \Rightarrow_i q)) \Rightarrow_i q$ is a tautology if and only if both $X$ and $Y$ have value $T$, that is, if and only if $i = 1$.

(b) Interpret your answer to part (a).

**Solution**: The conclusion is that if we want to define $p \Rightarrow q$ in a way that agrees with common sense whenever $p$ is true, and that allows us to apply the basic rule of inference modus ponens (if we know $p$, and we know $p \implies q$, then we can infer $q$), then we must declare $p \Rightarrow q$ to be true whenever $p$ is false.

5. Parentheses matter in general — "$\neg(p \wedge q)$" isn't the same as "$(\neg p) \wedge q$" — but presumably parentheses *don't* matter when a statement is just the "and" of a bunch of simpler statements. Indeed, it's easy to verify, via a truth table, the associative law that

$$p \wedge (q \wedge r) \quad \text{is the same as} \quad (p \wedge q) \wedge r.$$

What about the "and" of four statement, though? There are *five* different ways that parentheses can be put around $p \wedge q \wedge r \wedge s$, to make different-looking expressions:

A: $((p \wedge q) \wedge r) \wedge s$

B: $(p \wedge (q \wedge r)) \wedge s$

C: $p \wedge ((q \wedge r) \wedge s)$

D: $p \wedge (q \wedge (r \wedge s))$

E: $(p \wedge q) \wedge (r \wedge s)$.

These are all the same, right? Well, it should be pretty clear that A and B are the same: since $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ are the same (by associativity), it must be that $((p \wedge q) \wedge r) \wedge s$ and $(p \wedge (q \wedge r)) \wedge s$ are the same. Similarly, C and D are the same. But it's not immediately obvious that A/B is the same as C/D, or that E is the same as these two.

(a) Check that indeed B and C are the same. You could do this by building a truth table, but that is a drudge (16 rows!). There's a clever way to do it, that involves applying the associative rule in a slightly non-obvious way. Try to find the clever way!

**Solution**: Introduce three new statements, $P = p$, $Q = q \wedge r$ and $R = s$. Applying associativity to $P, Q, R$, that is, using

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R$$

says immediately

$$p \wedge ((q \wedge r) \wedge s) \text{ is equivalent to } (p \wedge (q \wedge r)) \wedge s,$$

which is exactly what we wanted to show.

(b) Check that B and E are the same. (Again, there's a slow way, and a clever way. Seek cleverness.)

**Solution**: We already know, from part a) above, and the discussion at the start of the question, that

$$(p \wedge (q \wedge r)) \wedge s \text{ is equivalent to } ((p \wedge q) \wedge r) \wedge s. \quad (\star)$$

Apply associativity to $((p \wedge q) \wedge r) \wedge s$, with "$p \wedge q$" playing the role of $p$, "$r$" playing the role of $q$, and "$s$" playing the role of $r$, to get

$$((p \wedge q) \wedge r) \wedge s \text{ is equivalent to } (p \wedge q) \wedge (r \wedge s). \quad (\star\star).$$

Combining $(\star)$ and $(\star\star)$ tells us that B and E are the same.

We will eventually be able to prove that for *any* $n$, no matter how the "and" of $n$ things is parenthesized, the result is logically the same. For the moment we will just take this for granted.

6. Later in the semester we will discuss the *Archimedean principle* of positive real numbers:

"If $N$ and $s$ are positive numbers, there's a positive number $t$ with $ts > N$."

(This is true no matter how big $N$ is or how small $s$ is.)

(a) If the universe of discourse for all the variables involved ($t, s$ and $N$) is the set of positive real numbers, then we can encode this statement as

$$(\forall N)(\forall s)(\exists t)(ts > N).$$

Write down the *negation* of the Archimedean principle symbolically, pulling the negation through all the quantifiers (your final answer should not involve the symbol "$\neg$"). Interpret the answer in ordinary English, by writing a sentence that captures what precisely it means for the Archimedean principle to *not* be true.

**Soloution**: Repeatedly using that negation flips the quantifier and negates the statement inside the quantifier, we get that

$$\neg((\forall N)(\forall s)(\exists t)(ts > N)) \text{ is equivalent to } (\exists N)(\exists s)(\forall t)\neg(ts > N).$$

The negation of "$ts > N$" is "$ts \leq N$", so

$$\neg((\forall N)(\forall s)(\exists t)(ts > N)) \text{ is equivalent to } (\exists N)(\exists s)(\forall t)(ts \leq N).$$

This says: The Archimedean principle not being true means that there are some positive numbers $N$ and $s$, such that for *all* positive numbers $t$, $t \times s$ is no bigger than $N$.

(b) If the universe of discourse for all the variables involved is the set of *all* real numbers (positive or otherwise), then the encoding of the principle is the slightly more complicated

$$(\forall N)(\forall s)\left[((N > 0) \wedge (s > 0)) \Rightarrow (\exists t)((t > 0) \wedge (ts > N))\right].$$

Negate this statement, pulling the negation all the way through all the quantifiers, and the implication. Your final answer should be a symbolic statement that has no "$\neg$" symbol in it, nor any "$\Rightarrow$".

**Soloution**: As in the last part, we quickly reach

$$(\exists N)(\exists s)\neg\left[((N > 0) \wedge (s > 0)) \Rightarrow (\exists t)((t > 0) \wedge (ts > N))\right]$$

We now need to negate the implication $((N > 0) \wedge (s > 0)) \Rightarrow (\exists t)((t > 0) \wedge (ts > N))$. Using that the negation of $p \Rightarrow q$ is $p \wedge (\neg q)$, we arrive at

$$(\exists N)(\exists s)\left[((N > 0) \wedge (s > 0)) \wedge \neg((\exists t)((t > 0) \wedge (ts > N)))\right].$$

This is equivalent to

$$(\exists N)(\exists s)\left[((N > 0) \wedge (s > 0)) \wedge (\forall t)(\neg((t > 0) \wedge (ts > N)))\right].$$

Now using DeMorgan's law, we reach

$$(\exists N)(\exists s)\left[((N > 0) \wedge (s > 0)) \wedge (\forall t)((\neg(t > 0)) \vee (\neg(ts > N)))\right].$$

which is equivalent to

$$(\exists N)(\exists s)\left[((N > 0) \wedge (s > 0)) \wedge (\forall t)((t \leq 0) \vee (ts \leq N))\right].$$

7. There is an equivalence underlying "proof by cases" (see Section 2.4 of the class notes):

$$(p_1 \vee p_2 \vee \cdots \vee p_k) \Rightarrow q \text{ is the same as } (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \cdots \wedge (p_n \Rightarrow q).$$

(Saying: if you want to prove that $p$ implies $q$, and $p$ breaks into cases, $p_1, p_2, \ldots, p_k$, then what you have to do is show that *each* of $p_1, p_2, \ldots, p_k$ on their own imply $q$.)

*Without using truth tables* show that this is a correct equivalence when $k = 2$, that is, show that

$$(p_1 \vee p_2) \Rightarrow q \text{ is the same as } (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q).$$

(What you need to do is use the various pairs of equivalent statements listed on page 18 of the class notes, to create a chain of statements, all of which are equivalent, with $(p_1 \vee p_2) \Rightarrow q$ at the start of the chain and $(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q)$ at the end.[1])

**Solution**: We use (in the first and last line below) that $a \Rightarrow b$ is equivalent to $(\neg a) \vee b$ (Implcation law) to get that

$$\begin{array}{rl} (p_1 \vee p_2) \Rightarrow q & \text{is equivalent to} \\ \neg(p_1 \vee p_2) \vee q & \text{which, by DeMorgan's law, is equivalent to} \\ ((\neg p_1) \wedge (\neg p_2)) \vee q & \text{which, by commutativity and distributivity, is equivalent to} \\ ((\neg p_1) \vee q) \wedge ((\neg p_2) \vee q) & \text{which is equivalent to} \\ (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q). & \end{array}$$

8. I have in mind a certain statement $P = P(p, q, r, s)$ that is made up of four simpler

[1] Here's an example of what I'm thinking of. Suppose you want to show that $(p \wedge (p \Rightarrow q)) \Rightarrow q$ is equivalent simply to $T$ (so, is a tautology), without using a truth table. In other words, you want to "reason out" that modus ponens is a valid logical inference, rather than "brute force" it. Here would be one possible approach:

$$\begin{array}{rcl} (p \wedge (p \Rightarrow q)) \Rightarrow q & \text{is equivalent to} & \neg(p \wedge (p \Rightarrow q)) \vee q \text{ (Implication law, or definition of implication)} \\ & \text{which is equivalent to} & ((\neg p) \vee \neg(p \Rightarrow q)) \vee q \text{ (De Morgan's law)} \\ & \text{which is equivalent to} & ((\neg p) \vee (p \wedge (\neg q))) \vee q \text{ (Negation of implication)} \\ & \text{which is equivalent to} & (((\neg p) \vee p) \wedge ((\neg p) \vee (\neg q))) \vee q \text{ (Distributive law)} \\ & \text{which is equivalent to} & ((p \vee (\neg p)) \wedge ((\neg p) \vee (\neg q))) \vee q \text{ (Commutative law)} \\ & \text{which is equivalent to} & (T \wedge ((\neg p) \vee (\neg q))) \vee q \text{ (Tautology law)} \\ & \text{which is equivalent to} & (((\neg p) \vee (\neg q)) \wedge T) \vee q \text{ (Commutative law)} \\ & \text{which is equivalent to} & ((\neg p) \vee (\neg q)) \vee q \text{ (Identity law)} \\ & \text{which is equivalent to} & (\neg p) \vee ((\neg q) \vee q) \text{ (Associative law)} \\ & \text{which is equivalent to} & (\neg p) \vee (q \vee (\neg q)) \text{ (Commutative law)} \\ & \text{which is equivalent to} & (\neg p) \vee T \text{ (Tautology law)} \\ & \text{which is equivalent to} & T \text{ (Domination law).} \end{array}$$

This seems like overkill: it would have been much faster, *in this particular case*, to use a truth table. But as the number of simpler propositions involved grows, the truth table approach becomes less and less desirable. For example, with two propositions (as we have here) the truth table has only $2^2 = 4$ rows; but with 20 propositions, the truth table has $2^20 \approx 1,000,000$ rows! At that point, it is completely impractical to use a truth table to verify an equivalence, and it is absolutely necessary to the pure reasoning illustrated in the example above, and asked for in this homework problem.

statements, $p, q, r$ and $s$. Here's the truth table of $P$:

| $p$ | $q$ | $r$ | $s$ | $P$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

Express $P$ in terms of $p, q, r$ and $s$, using the symbols $\neg$, $\vee$ and $\wedge$. (This is closely related to Exercise number 5 on the first tutorial handout.)

**Solution**: We want a statement which is true *either* when $p, q$ and $s$ are true and $r$ is false (third line of truth table), *or* when $p$ and $q$ are true and $r$ and $s$ are false (fourth line), *or* when $p$ and $r$ are true and $q$ and $s$ are false (sixth line), *or*, et cetera. The statement

$$p \wedge q \wedge (\neg r) \wedge s$$

is true *exactly* when when $p, q$ and $s$ are true and $r$ is false; the statement

$$p \wedge q \wedge (\neg r) \wedge (\neg s)$$

is true *exactly* when $p$ and $q$ are true and $r$ and $s$ are false, et cetera. So one way to get a statement with the required truth table is to take each of the 7 "ands" of combinations of $p$, $\neg p$, $q$, $\neg q$, etc., corresponding to lines in the truth table with "$T$" in the final column, and "or" those together. This yields the (somewhat long) statement:

$$\begin{array}{cc} & (p \wedge q \wedge (\neg r) \wedge s) \\ \vee & (p \wedge q \wedge (\neg r) \wedge (\neg s)) \\ \vee & (p \wedge (\neg q) \wedge r \wedge (\neg s)) \\ \vee & (p \wedge (\neg q) \wedge (\neg r) \wedge s) \\ \vee & ((\neg p) \wedge q \wedge r \wedge s) \\ \vee & ((\neg p) \wedge q \wedge r \wedge (\neg s)) \\ \vee & ((\neg p) \wedge (\neg q) \wedge r \wedge s). \end{array}$$

For each row of the truth table that ends in "$T$", exactly one of the above and-ed expressions is true, so the disjunction is true; for all other rows, *none* of the above and-ed expressions is true, so the disjunction is false. So the above expression has the required truth-table.[2]

From this example, you should be able to convince yourself of the fact, alluded to in the first tutorial worksheet (Exercise 5), that *any* truth table can be realized using $\neg$, $\wedge$ and $\vee$.

9. This question concerns the statement $(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$. This may seem somewhat random, but it is in fact (one special case of) and incredibly important logical operation in theoretical computer science. I won't say the name commonly given to the operation, as it gives the following question away.

   (a) Write down the truth table of the statement

   $$(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s).$$

   **Solution**: Really? But it has 16 rows! Oh, ok. Here it is, without any intermediate columns:

| $p$ | $q$ | $r$ | $s$ | $(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

   (b) There is a *very* simple description of what $(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$ is doing. Can you find it? Your answer should take the form:

   "$(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$ is true exactly when ..."

   and you should not use more than eleven words to replace "..." (with $p$, $q$, $r$ and $s$ each counting as one word).

   **Solution**:

   "$(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$ is true exactly when *an even number of* $p, q, r$ *and* $s$ *are true.*"

[2]If you answered the question in this way, you discovered *Disjunctive Normal Form* — `https://en.wikipedia.org/wiki/Disjunctive_normal_form`.

(11 words!)

$(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$ is an example of a *parity function*, and is important in theoretical computer science, see e.g. `https://en.wikipedia.org/wiki/Parity_function`.