Math 10850, Honors Calculus 1

Homework 10

Solutions

1. Remember that for a function f and a number L, "$\lim_{x\to\infty} f(x) = L$" means that

   for all $\varepsilon > 0$ there is M such that for all x, if $x > M$ then $|f(x) - L| < \varepsilon$.

   Implicit in this definition is that f is "defined near ∞": there is N such that the domain of f includes (N, ∞).

   (a) Prove that if $\lim_{x\to\infty} f(x) = L$ and $L \neq 0$ then

   i. the function $1/f$ is actually defined near ∞, and

   Solution: By definition (taking $\varepsilon = |L|/2$) there is an N such that $x > N$ implies $|f(x) - L| < |L|/2$, or equivalently $L - |L|/2 < f(x) < L + |L|/2$.
   If $L > 0$, then the lower bound says that for $x > N$, $f(x) > L/2 > 0$, so (N, ∞) is in the domain of $1/f$; while if $L < 0$, then the upper bound says that for $x > N$, $f(x) < L/2 < 0$, so (N, ∞) is in the domain of $1/f$. Either way, $1/f$ is defined near infinity.

   ii. $\lim_{x\to\infty} 1/f(x)$ exists and equals $1/L$.[1]

   Solution: From the previous part, we know that $1/f$ is defined near ∞.
   Let $\varepsilon > 0$ be given. We want to find M such that $x > M$ implies

   $$\left|\frac{1}{f(x) - \frac{1}{L}}\right| < \varepsilon.$$

   Now

   $$\left|\frac{1}{f(x) - \frac{1}{L}}\right| = \frac{|f(x) - L|}{|f(x)||L|}.$$

   We know (from the previous part) that there is N such that $x > N$ implies $|f(x)| > |L|/2$. We also know (from the hypothesis $f \to L$ near ∞) that there is N' such that $x > N$ implies $|f(x) - L| < \varepsilon |L|^2/2$. For $x > \max\{N, N'\}$ we have

   $$\left|\frac{1}{f(x) - \frac{1}{L}}\right| = \frac{|f(x) - L|}{|f(x)||L|} < \frac{\varepsilon |L|^2/2}{|L|^2/2} = \varepsilon.$$

[1] Here I want a complete and correct proof, laid out in a readable way. You have a hypothesis, namely $\lim_{x\to\infty} f(x) = L$, that is equivalent to some precise ε-M statement, and you have a conclusion that you want to reach, namely $\lim_{x\to\infty} 1/f(x) = 1/L$, that is also equivalent to some ε-M statement. You should argue the validity of the conclusion, using the hypothesis, and coherent logic. It will be helpful, perhaps, to review the proof of the sum/product/reciprocal theorem for ordinary limits (particularly the reciprocal part), before doing this question.

Since $\varepsilon > 0$ was arbitrary, this shows that $\lim_{x\to\infty} 1/f(x)$ exists and equals $1/L$.

(b) Carefully prove that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to 0^+} f(1/x) = L$.[2]

Solution: Let $\varepsilon > 0$ be given. We want $\delta > 0$ such that $0 < x < \delta$ implies $|f(1/x) - L| < \varepsilon$. We know that there is N such that $y > N$ implies $|f(y) - L| < \varepsilon$. Just to make sure that we are working with positive numbers, take $M = \max\{N, 1\}$; we have that $y > M$ implies

$$|f(y) - L| < \varepsilon. \quad (\star)$$

Take $\delta = 1/M$ (note $\delta > 0$ by choice of M). If $0 < x < \delta$ then $1/x > 1/\delta = M$, so, by $(\star)$, we have $|f(1/x) - L| < \varepsilon$.

This is what we wanted of δ; so we have shown that indeed $\lim_{x\to 0^+} f(1/x) = L$.

(c) **OPTIONAL!** If you want more practice at infinite limits/limits at infinity, look at Spivak (4th edition), Chapter 5, problems 32 through 40. None of these will be graded. Ideally, you should just be able to look at the functions and "sense" what the limits at infinity are, using the same techniques that we use to "sense" the limits of continuous functions as they approach finite values. Check your answers by plotting some of the graphs using your favorite plotting tool.

2. **OPTIONAL!** Residents of "$\mathbb{Q}$-world" stopped constructing the number system at axiom P12; as far as they are concerned, the only numbers are rational numbers. Show that Rolle's theorem is *false* in $\mathbb{Q}$-world: give a $\mathbb{Q}$-world example of a continuous function $f : [a, b] \to \mathbb{Q}$ that is differentiable on (a, b), that has $f(a) = f(b)$, but for which there is *no* $c \in (a, b)$ with $f'(c) = 0$.

3. In class we proved the *second derivative test*: if f is defined at and near x, and if $f'(x) = 0$ and $f''(x) > 0$, then f has a local minimum at x.

(a) Prove the following partial converse:

> "Suppose f is defined at and near a, and that $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$."

Solution: f is differentiable at a (it's twice differentiable there, so certainly differentiable), and f has a local minimum at a, so by Fermat principle we have $f'(a) = 0$.

Suppose, for a contradiction, that $f''(a) < 0$. We can apply the second derivative test to f at a, to conclude that f has a local maximum at a (here we are using what we derived above, that $f'(a) = 0$; without that we couldn't use the second derivate test). But if f has a local maximum at a, and a local minimum, there must be some interval around a on which f is constant; so on that interval, f' must be constantly 0, so $f''(a) = 0$, a contradiction!

We conclude that $f''(a) \geq 0$.

[2]One could also prove that if $\lim_{x\to 0^+} f(1/x) = L$ then $\lim_{x\to\infty} f(x) = L$. So limits at infinity can be translated to ordinary limits.

(b) Did your proof use anything that relies on the completeness axiom?

Solution: I used the second derivative test, whose proof used the first derivative test, whose proof used that positive derivative implies increasing, whose proof used MVT, whose proof used EVT, whose proof used the completeness axiom. So the proof that I gave above does require completeness. (This doesn't say that all proofs of the converse to the second derivative test need completeness; I don't at the moment know whether that is true.)

(c) Is the following "full converse" of the second derivative test true? (Justify your answer!)

> "Suppose f is defined at and near a, and that $f''(a)$ exists. If f has a local minimum at a, then $f''(a) > 0$."

Solution: This is *false* — $f(x) = x^4$ at 0 provides a counterexample.

4. (a) Let $a_1 < a_2 < \cdots < a_n$. Find the minimum value of $f(x) = \sum_{i=1}^{n}(x - a_i)^2$, and the x at which the minimum occurs.

Solution: We have

$$f'(x) = \sum_{i=1}^{n} 2(x - a_i) = 2nx - 2\sum_{i=1}^{n} a_i.$$

Notice that $f''(x) = 2n$, which is positive for all x; so any point at which $f'(x) = 0$, by the second derivative test x is a local minimum. There is only one point at which $f'(x) = 0$, namely

$$x^\star = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

So f has a local min at $x^\star$; but it is its global minimum there, since:

- $f'(x) > 0$ for $x > x^\star$, so (by corollary of MVT) f in increasing on $[x^\star, \infty)$ and strictly increasing on $(x^\star, \infty)$, so for $x > x^\star$ we have $f(x) > f(x') \geq f(x^\star)$, where x' is any number in $(x^\star, x)$, and
- $f'(x) < 0$ for $x < x^\star$, so (by corollary of MVT) f in decreasing on $(-\infty, x^\star]$ and strictly decreasing on $(-\infty, x^\star)$, so for $x < x^\star$ we have $f(x) > f(x') \geq f(x^\star)$, where x' is any number in $(x, x^\star)$.

So the minimum value of f is at $x^\star$ (which happens to be the average of the a_i's), and is

$$\sum_{i=1}^{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) - a_i\right)^2.$$

Notes:

i. Apparently, the instruction $a_1 < a_2 < \cdots < a_n$ was a red herring.

ii. Embedded in this solution is a general principle, that will be useful for other questions on this set:

Suppose $f : (a, b) \to \mathbb{R}$ is differentiable everywhere, and that there is some $c \in (a, b)$ with

- $f'(c) = 0$,
- $f' > 0$ on (c, b),
- $f' < 0$ on (a, c).

Then f has its unique global minimum at c. If instead

- $f'(c) = 0$,
- $f' < 0$ on (c, b),
- $f' > 0$ on (a, c),

then f has its unique global minimum at c. For both of these, the same result holds if a is replaced by $-\infty$, b by $+\infty$, or both.

(b) **OPTIONAL!** Let $a > 0$. Find the maximum value of $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x+a|}$. (To compute the derivative of f, it might be helpful to consider separately the intervals $(-\infty, -a]$, $[-a, 0]$ and $[0, \infty)$.)

5. Find, among all right circular cylinders of fixed volume $V > 0$, the one with the smallest surface area (include the areas of the top and bottom circular faces; see the figure below marked "Figure 24").

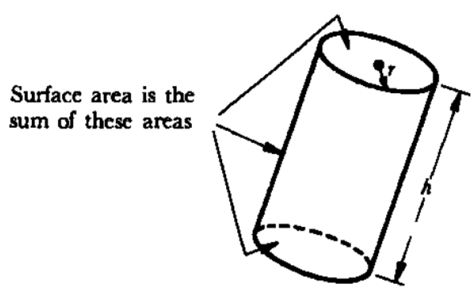

FIGURE 24

Solution: Let r be the radius of the cylinder, and h the height. We have that r and h are related by

$$V = \pi r^2 h,$$

and that the surface area is

$$S = 2\pi r^2 + 2\pi r h.$$

This is a function of two variables, but we can make if a function of just one variable by using $V = \pi r^2 h$ to eliminate either h or r. Since r as a function of h involves a square root, it seems reasonable to instead express h as a function of r:

$$h = \frac{V}{\pi r^2},$$

yielding

$$S(r) = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

We want to minimize this, as r varies over $(0, \infty)$ (r can be made arbitrarily small while the volume stays at V — this requires h to be made arbitrarily large — and conversely r can be made arbitrarily large while the volume stays at V — this requires h to be made arbitrarily small. But r cannot take the value 0, as this forces the volume to be 0).

We have

$$S'(r) = 4\pi r - \frac{2V}{r^2}.$$

This equals 0 when $r^3 = \frac{V}{2\pi}$ or

$$r = \sqrt[3]{\frac{V}{2\pi}} := r^\star.$$

The derivative is positive on $(r^\star, \infty)$ and negative on $(0, r^\star)$, so by the general result discussed in a previous question, the surface area has its unique global minimum at radius $r^\star = \sqrt[3]{\frac{V}{2\pi}}$. At this point the height is

$$h^\star = \frac{V}{\pi \left(\sqrt[3]{\frac{V}{2\pi}}\right)^2}.$$

After you have finished solving the problem, complete the following slogan in six words or less:

> "among all right circular cylinders of fixed volume, the one with the smallest total surface area has ..."

Solution: It doesn't seem easy to concisely express the solution! But, if we simplify the optimal height, we get

$$h^\star = \frac{V}{\pi \left(\sqrt[3]{\frac{V}{2\pi}}\right)^2} = \frac{2^{2/3}V^{1/3}}{\pi^{1/3}} = 2\left(\frac{V^{1/3}}{2^{1/3}\pi^{1/3}}\right) = 2r^\star.$$

Now we can finish the slogan in six words:

> "among all right circular cylinders of fixed volume, the one with the smallest total surface area has height equal to twice its radius".

We could have gotten away with *four* words:

> "among all right circular cylinders of fixed volume, the one with the smallest total surface area has equal height and diameter".

6. You have to cross a circular lake of radius 1 mile. You can row at 2 miles per hour, and walk at w miles per hour. (See the figure below marked "Figure 28".) What route should you choose, to minimize the time spent crossing the lake? (Presumably, the answer that you get depends on w — you may assume $w > 0$).

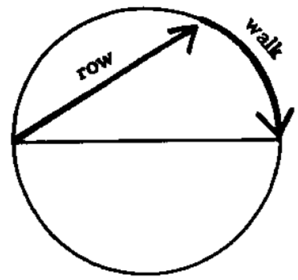

FIGURE 28

Solution: Label the starting point A, the finishing point B, the point where the rowing stops and the walking begins C (assuming, without any loss of generality, that C is on the top half of the lake), and the center of the lake O. Suppose that the angle the rowboat makes with the diameter of the lake is t; that is, angle CAB is t. The range of possible values for t is from 0 (row straight across lake) to $\pi/2$ (walk all the way around).

Because OAC is equilateral with OA equaling OC, angle AOC is t. Because angles made in a semicircle are right angles, angle OCB is $\pi/2 - t$. Because OCB is equilateral with OC equaling OB, angle COB is $2t$.

Because ACB is a right angle, we have $\cos t$ equals half AC, or $AC = 2\cos t$; this is the distance rowed (here using that the diameter of the lake is 2). Because the circumference of the lake is 2π, the distance walked is $(2t)/(2\pi)$ proportion of 2π, or $2t$.

Using distance equals speed by time, the time spent rowing is $(2\cos t)/2$ or $\cos t$, and the time spent walking is $2t/w$. So the total time spent traveling is

$$f(t) = \cos t + 2t/w.$$

Our goal is to minimize this continuous, differentiable function on the closed interval $[0, \pi/2]$; we know that to do this, all we need do is compare the function values at the endpoints (0 and $\pi/2$) and at any critical points (points of zero derivative) there might be in the interval. We have

$$f(0) = 1 \quad \text{(row all the way across — 1 hour)}$$

and

$$f(\pi/2) = \frac{\pi}{w} \quad \text{(walk all the way around — } \pi/w \text{ hours).}$$

We also have to check the transit time at points where the derivative is 0. We have

$$f'(t) = -\sin t + 2/w$$

which equals 0 when $\sin t = 2/w$. For $w < 2$, there is no such t (sin only takes values between -1 and 1).

For $w \geq 2$, there is one value of t at which $\sin t = 2/w$, namely $t = \sin^{-1}(2/w)$. It seems like it will quite awkward to compare the transit time at $t = \sin^{-1}(2/w)$ (which is

$$f(\sin^{-1}(2/w)) = \cos(\sin^{-1}(2/w)) + \frac{2\sin^{-1}(2/w)}{w})$$

to that at $t = 0, \pi/2$. But it fact, we don't need to! We have

$$f''(t) = -\cos t$$

which is always *negative* on the interval $(0, \pi/2)$, so whatever the transit time is at the critical point, it a local *max*, not a local min, and if fact, because $f'(t) > 0$ for $t < \sin^{-1}(2/w)$ and $f'(t) < 0$ for $t > \sin^{-1}(2/w)$ we get (similar to previous questions) that the transit time at the critical point is a global max. So in considering the fastest crossing time, we get to ignore this critical point.

So, for *all* w, the fastest transit time is

$$\min\{1, \pi/w\}.$$

Evidently this is 1 if $w \leq \pi$ (row straight across) and π/w if $w \geq \pi$ (walk all the way around).

So, in summary:

- if $w < \pi$ you should only row;
- if $w > \pi$ you should only walk;
- if $w = \pi$ you could either row of walk (but not a mixture).

7. **OPTIONAL!** Suppose that $f'(x) \geq M > 0$ for all $x \in [0, 1]$. Show that there is an interval of length $1/4$ on which $|f| > M/4$.

8. (a) Suppose that $f'(x) > g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) > g(x)$ for all $x > a$ and that $f(x) < g(x)$ for all $x < a$.

   Solution: Consider the function h defined by $f - g$. We have $h'(x) > 0$ for all x, and $h(a) = 0$.
   For $x > a$, by the MVT there is $c \in (a, x)$ with

   $$h'(c) = \frac{h(x) - h(a)}{x - a} = \frac{h(x)}{x - a}.$$

   Now $h'(c) > 0$ and $x - a > 0$, so $h(x) > 0$, or $f(x) > g(x)$.
   For $x < a$, by the MVT there is $c \in (x, a)$ with

   $$h'(c) = \frac{h(a) - h(x)}{a - x} = \frac{-h(x)}{a - x}.$$

   Now $h'(c) > 0$ and $a - x > 0$, so $-h(x) > 0$, so $h(x) < 0$ or $f(x) < g(x)$.

   (b) Suppose that $f'(x) \geq g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) \geq g(x)$ for all $x \geq a$.

   Solution: Essentially the same as the previous part: consider the function h defined by $f - g$. We have $h'(x) \geq 0$ for all x, and $h(a) = 0$.

For $x = a$, certainly $f(x) \geq g(x)$ (in fact they are equal). For $x > a$, by the MVT there is $c \in (a, x)$ with

$$h'(c) = \frac{h(x) - h(a)}{x - a} = \frac{h(x)}{x - a}.$$

Now $h'(c) \geq 0$ and $x - a > 0$, so $h(x) \geq 0$, or $f(x) \geq g(x)$.

(c) Suppose that $f'(x) \geq g'(x)$ for all x, and that $f(a) = g(a)$. Suppose also that $f'(x_0) > g'(x_0)$ for some $x_0 > a$. Show that $f(x) > g(x)$ for all $x \geq x_0$.

Solution: As in the previous two parts, we translate to a question about the function $h = f - g$. We have

- $h'(x) \geq 0$ for all x
- $h(a) = 0$ and
- $h'(x_0) > 0$ for some $x_0 > a$,

and we would like to conclude that $h(x) > 0$ for all $x \geq x_0$.

Suppose $h(x_0) < 0$. Then by the MVT applied to $[a, x_0]$, at some point in the interval (a, x_0) we would have negative derivative, a contradiction.

Suppose $h(x_0) = 0$. Since $h'(x) \geq 0$ for all x, it follows that h is weakly increasing on $[a, x_0]$, so, since $h(a) = h(x_0) = 0$, it must be that h is identically 0 on $[a, x_0]$. But that says that the derivative of h at x_0 from below is 0, contradicting $h'(x_0) > 0$.

So, we get that $h(x_0) > 0$, say $h(x_0) = \kappa > 0$. Since $h'(x) \geq 0$ for all x, it follows that h is weakly increasing on $[x_0, \infty]$, so $h(x) \geq \kappa > 0$ for all $x \geq x_0$.

9. **OPTIONAL!** For a real number m, define $f_m : \mathbb{R} \to \mathbb{R}$ by $f_m(x) = x^3 - 3x + m$. Prove that it is not possible for f_m to have two distinct roots in $[0, 1]$ (that is, prove that no matter what value m takes, it is not possible to find $x_1 \neq x_2 \in [0, 1]$ with $f(x_1) = f(x_2) = 0$).

10. Define a function f by $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(a) Verify, *using an* ε-δ argument, that f is continuous at 0 (and so it is continuous at all reals). [This should go quickly if you judiciously use $|\sin 1/x| \leq 1$.]

Solution: Given $\varepsilon > 0$, take $\delta = \sqrt{\varepsilon}$. If $|x| < \delta$ then

$$\left|x^2 \sin(1/x) - 0\right| = |x|^2 \left|\sin(1/x) - 0\right| \leq |x|^2 < \delta^2 = \varepsilon.$$

This shows that $\lim_{x \to 0} x^2 \sin(1/x) = 0$, so f is continuous at 0.

(b) *Using the definition of the derivative*, verify that f is differentiable at 0 (and so it is differentiable at all reals).

Solution: We have

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{h^2 \sin(1/h) - 0}{h} \\ &= h \sin(1/h), \end{aligned}$$

and this (as we have seen) tends to 0 as $h \to 0$; so $f'(0)$ exists and equals 0.

(c) Show that f' is *not* continuous at 0.

Solution: We have (using chain rule, et cetera, and the result of part (a)) that $f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. It cannot be that $2x\sin(1/x) - \cos(1/x) \to 0$ as $x \to 0$. If that was true, then, since $2x\sin(1/x) \to 0$ as $x \to 0$, we would also get (via the sum theorem for limits) that $\cos(1/x) \to 0$ as $x \to 0$; but in fact $\cos(1/x)$ does not approach a limit as x approaches 0 (it oscillates infinitely between 1 and -1, as $\sin(1/x)$ does). So $f'(0)$ does not exist.

11. A function f defined on an interval I (open, closed, mixed, infinite, ...) is said to have the *intermediate value property* if for any $a < b \in I$, f takes on all values between $f(a)$ and $f(b)$ on the interval $[a, b]$. So the Intermediate Value Theorem (in its most general form) says that if f is continuous on I[3] then f has the intermediate value property.

It's not obvious that derivatives have the intermediate value property — that is, if f is differentiable on an interval I[4] then it is not obvious that f' has the intermediate value property. The issue is that f' may not be continuous on I (see the previous question for an example), so we cannot use IVT to deduce the intermediate value property.

Nevertheless, derivatives *do* have the intermediate value property. This question brings you through a proof of this quite remarkable fact.

(a) Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) > 0 > f'(b)$.

i. Explain clearly why f has a maximum point on $[a, b]$ (this should be very brief — appeal to a powerful theorem we have proven, after explaining why f satisfies the hypothesis of the theorem).

Solution: f is differentiable on $[a, b]$, so continuous, so by EVT it has a maximum point.

ii. Prove that neither a nor b can be a maximum point of f on $[a, b]$ (use the definition of the (one-sided) derivative).

Solution: Suppose that a is the maximum point. Then for any $h > 0$ we have

$$\frac{f(a+h) - f(a)}{h} \leq 0,$$

so $f'(a) = f'_+(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \leq 0$, contradicting the hypothesis $f'(a) > 0$. Similarly, suppose that b is the maximum point. Then for any $h < 0$ we have

$$\frac{f(b+h) - f(b)}{h} \geq 0,$$

[3] Here"continuous" is understood to mean "continuous from above" if we happen to be at the left end-point of I (if I has such a point), and "continuous from below" if we happen to be at the right end-point of I (if I has such a point).

[4] Here"differentiable" is understood to mean "differentiable from above" if we happen to be at the left end-point of I (if I has such a point), and "differentiable from below" if we happen to be at the right end-point of I (if I has such a point).

so $f'(b) = f'_-(b) = \lim_{h \to 0} \frac{f(b+h)-f(b)}{h} \geq 0$, contradicting the hypothesis $f'(b) < 0$.

iii. The previous part shows that if c is a maximum point of f on $[a, b]$, then $c \in (a, b)$. Explain why $f'(c) = 0$ (this should be very brief — appeal to a result we have proven, after explaining why f satisfies the hypothesis of the result)

Solution: The Fermat principle says that if $g : (a, b) \to \mathbb{R}$ has a maximum point at $c \in (a, b)$, and g is differentiable at c, then $g'(c) = 0$. Here f is certainly differentiable at c, since it is differentiable on all of (a, b), so the Fermat principle applies to let us conclude $f'(c) = 0$.

(b) **OPTIONAL!** Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) < 0 < f'(b)$. Show that there is $c \in (a, b)$ with $f'(c) = 0$. (This is almost identical to part (a) — it just involves changing one word in the guide above, and a few inequalities in the proof).

(c) Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) \neq f'(b)$. Let y be a number that lies strictly between $f'(a)$ and $f'(b)$.

i. Find a function g defined on I, that is differentiable on all of I, for which exactly one of $g'(a), g'(b)$ is negative, the other one is positive, and for which if $g'(c) = 0$ then $f'(c) = y$. (You should build g from f, in a fairly simple way — think about how we derived MVT from Rolle's theorem).

Solution: Let $g(x) = f(x) - xy$. We have that g defined on I and differentiable on all of I. We have $g'(x) = f'(x) - y$, so $g'(a) = f'(a) - y$ and $g'(b) = f'(b) - y$. So if $f'(a) > y > f'(b)$ then $g'(a) > 0 > g'(b)$, while if $f'(a) < y < f'(b)$ then $g'(a) < 0 < g'(b)$. Finally, if $g'(c) = 0$ then $0 = f'(c) - y$, or $f'(c) = y$.

ii. By applying the results of the first two parts of this question to g, prove that there is $c \in (a, b)$ with $f'(c) = y$.

Solution: By the first two parts, there is $c \in (a, b)$ with $g'(c) = 0$, so $f'(c) = y$.

(d) Give an example of a function defined on all reals, that cannot possible arise as the derivative of some other function.

Solution: Any function with a "jump" discontinuity (and so fails to have the intermediate value property); e.g.

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$