Math 10850, Honors Calculus 1

Homework 10

Due in class Friday December 6

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

This homework covers infinite limits/limits at infinity, and some applications of the derivative. You should read Section 6.6 of the class notes (limits), Section 8.6 (chain rule), and Sections 9.1 and 9.2 (maxima/minima/Mean Value Theorem).

Assignment

1. Remember that for a function f and a number L, "$\lim_{x\to\infty} f(x) = L$" means that

   for all $\varepsilon > 0$ there is M such that for all x, if $x > M$ then $|f(x) - L| < \varepsilon$.

   Implicit in this definition is that f is "defined near ∞": there is N such that the domain of f includes (N, ∞).

   (a) Prove that if $\lim_{x\to\infty} f(x) = L$ and $L \neq 0$ then
      i. the function $1/f$ is actually defined near ∞, and
      ii. $\lim_{x\to\infty} 1/f(x)$ exists and equals $1/L$.[1]

   (b) Carefully prove that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to 0^+} f(1/x) = L$.[2]

   (c) **OPTIONAL!** If you want more practice at infinite limits/limits at infinity, look at Spivak (4th edition), Chapter 5, problems 32 through 40. None of these will be graded. Ideally, you should just be able to look at the functions and "sense" what the limits at infinity are, using the same techniques that we use to "sense" the limits of continuous functions as they approach finite values. Check your answers by plotting some of the graphs using your favorite plotting tool.

[1] Here I want a complete and correct proof, laid out in a readable way. You have a hypothesis, namely $\lim_{x\to\infty} f(x) = L$, that is equivalent to some precise ε-M statement, and you have a conclusion that you want to reach, namely $\lim_{x\to\infty} 1/f(x) = 1/L$, that is also equivalent to some ε-M statement. You should argue the validity of the conclusion, using the hypothesis, and coherent logic. It will be helpful, perhaps, to review the proof of the sum/product/reciprocal theorem for ordinary limits (particularly the reciprocal part), before doing this question.

[2] One could also prove that if $\lim_{x\to 0^+} f(1/x) = L$ then $\lim_{x\to\infty} f(x) = L$. So limits at infinity can be translated to ordinary limits.

2. **OPTIONAL!** Residents of "$\mathbb{Q}$-world" stopped constructing the number system at axiom P12; as far as they are concerned, the only numbers are rational numbers. Show that Rolle's theorem is *false* in $\mathbb{Q}$-world: give a $\mathbb{Q}$-world example of a continuous function $f : [a, b] \to \mathbb{Q}$ that is differentiable on (a, b), that has $f(a) = f(b)$, but for which there is *no* $c \in (a, b)$ with $f'(c) = 0$.

3. In class we proved the *second derivative test*: if f is defined at and near x, and if $f'(x) = 0$ and $f''(x) > 0$, then f has a local minimum at x.

   (a) Prove the following partial converse:

   > "Suppose f is defined at and near a, and that $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$."

   (b) Did your proof use anything that relies on the completeness axiom?

   (c) Is the following "full converse" of the second derivative test true? (Justify your answer!)

   > "Suppose f is defined at and near a, and that $f''(a)$ exists. If f has a local minimum at a, then $f''(a) > 00$."

4. (a) Let $a_1 < a_2 < \cdots < a_n$. Find the minimum value of $f(x) = \sum_{i=1}^{n}(x - a_i)^2$, and the x at which the minimum occurs.

   (b) **OPTIONAL!** Let $a > 0$. Find the maximum value of $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x+a|}$. (To compute the derivative of f, it might be helpful to consider separately the intervals $(-\infty, -a]$, $[-a, 0]$ and $[0, \infty)$.)

5. Find, among all right circular cylinders of fixed volume $V > 0$, the one with the smallest surface area (include the areas of the top and bottom circular faces; see the figure below marked "Figure 24").

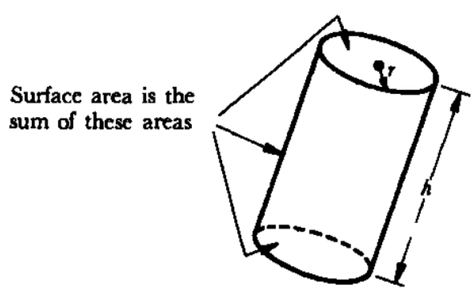

FIGURE 24

After you have finished solving the problem, complete the following slogan in six words or less:

> "among all right circular cylinders of fixed volume, the one with the smallest total surface area has ..."

6. You have to cross a circular lake of radius 1 mile. You can row at 2 miles per hour, and walk at w miles per hour. (See the figure below marked "Figure 28".) What route should you choose, to minimize the time spent crossing the lake? (Presumably, the answer that you get depends on w — you may assume $w > 0$).

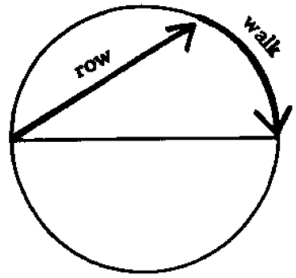

FIGURE 28

7. **OPTIONAL!** Suppose that $f'(x) \geq M > 0$ for all $x \in [0,1]$. Show that there is an interval of length $1/4$ on which $|f| > M/4$.

8. (a) Suppose that $f'(x) > g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) > g(x)$ for all $x > a$ and that $f(x) < g(x)$ for all $x < a$.
   (b) Suppose that $f'(x) \geq g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) \geq g(x)$ for all $x \geq a$.
   (c) Suppose that $f'(x) \geq g'(x)$ for all x, and that $f(a) = g(a)$. Suppose also that $f'(x_0) > g'(x_0)$ for some $x_0 > a$. Show that $f(x) > g(x)$ for all $x \geq x_0$.

9. **OPTIONAL!** For a real number m, define $f_m : \mathbb{R} \to \mathbb{R}$ by $f_m(x) = x^3 - 3x + m$. Prove that it is not possible for f_m to have two distinct roots in $[0,1]$ (that is, prove that no matter what value m takes, it is not possible to find $x_1 \neq x_2 \in [0,1]$ with $f(x_1) = f(x_2) = 0$).

10. Define a function f by $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.
    (a) Verify, *using an ε-δ argument*, that f is continuous at 0 (and so it is continuous at all reals). [This should go quickly if you judiciously use $|\sin 1/x| \leq 1$.]
    (b) *Using the definition of the derivative*, verify that f is differentiable at 0 (and so it is differentiable at all reals).
    (c) Show that f' is *not* continuous at 0.

11. A function f defined on an interval I (open, closed, mixed, infinite, ...) is said to have the *intermediate value property* if for any $a < b \in I$, f takes on all values between $f(a)$ and $f(b)$ on the interval $[a,b]$. So the Intermediate Value Theorem (in its most general form) says that if f is continuous on I[3] then f has the intermediate value property.

[3] Here "continuous" is understood to mean "continuous from above" if we happen to be at the left end-point of I (if I has such a point), and "continuous from below" if we happen to be at the right end-point of I (if I has such a point).

It's not obvious that derivatives have the intermediate value property — that is, if f is differentiable on an interval I[4] then it is not obvious that f' has the intermediate value property. The issue is that f' may not be continuous on I (see the previous question for an example), so we cannot use IVT to deduce the intermediate value property.

Nevertheless, derivatives *do* have the intermediate value property. This question brings you through a proof of this quite remarkable fact.

(a) Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) > 0 > f'(b)$.

   i. Explain clearly why f has a maximum point on $[a, b]$ (this should be very brief — appeal to a powerful theorem we have proven, after explaining why f satisfies the hypothesis of the theorem).

   ii. Prove that neither a nor b can be a maximum point of f on $[a, b]$ (use the definition of the (one-sided) derivative).

   iii. The previous part shows that if c is a maximum point of f on $[a, b]$, then $c \in (a, b)$. Explain why $f'(c) = 0$ (this should be very brief — appeal to a result we have proven, after explaining why f satisfies the hypothesis of the result)

(b) **OPTIONAL!** Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) < 0 < f'(b)$. Show that there is $c \in (a, b)$ with $f'(c) = 0$. (This is almost identical to part (a) — it just involves changing one word in the guide above, and a few inequalities in the proof).

(c) Let f be a function defined on some interval I, that is differentiable on all of I. Let a, b be some numbers in I, with $a < b$. Suppose that $f'(a) \neq f'(b)$. Let y be a number that lies strictly between $f'(a)$ and $f'(b)$.

   i. Find a function g defined on I, that is differentiable on all of I, for which exactly one of $g'(a), g'(b)$ is negative, the other one is positive, and for which if $g'(c) = 0$ then $f'(c) = y$. (You should build g from f, in a fairly simple way — think about how we derived MVT from Rolle's theorem).

   ii. By applying the results of the first two parts of this question to g, prove that there is $c \in (a, b)$ with $f'(c) = y$.

(d) Give an example of a function defined on all reals, that cannot possible arise as the derivative of some other function.

[4] Here "differentiable" is understood to mean "differentiable from above" if we happen to be at the left end-point of I (if I has such a point), and "differentiable from below" if we happen to be at the right end-point of I (if I has such a point).