Math 10850, Honors Calculus 1

Homework 1

Due in class Friday September 6

General notes on the homework

The purpose of the weekly homework set is to get you to engage with the definitions and theorems/proofs/results that we are studying in class. Ideally, you should find some of the homework problems to be fairly easy, the majority to be somewhat tricky, and a few to be utterly confounding. You should *not* by worried if some problems confound you — these problems are intended to get you to think hard about the topic in question, and through thinking, talking, and experimenting, begin to internalize the topic. My hope is that after some work on your part, confusion will be replaced by clarity.

I encourage you to talk to each other about the homework problems. Take the time to gather in small groups and exchange ideas. Mathematics is a subject of ideas, and ideas are not valuable unless they can be clearly communicated. The earlier you get into the habit of expressing you ideas out loud to other people, the easier this course will be.

You have resources at your disposal to help you with homework. In particular,

- I have office hours,
- the TA has office hours,
- there is a weekly tutorial, during which there will be a chance to talk about the homework.

I strongly encourage you to make use of these resources. But I also encourage you to give genuine thought to the problems that you need help with, before coming to office hours. Ideally you should be coming to me or the TA saying "I tried approach to question Y, and encountered the following issue at point Z. I don't know how exactly to get past this issue", rather than saying "How do I do question Y?"!

One more resource that I must highlight is the math bunker, a room in the library (in the basement of Hayes-Healy) that for two hours a night, Sunday through Thursday, is staffed by upperclass honors math majors. The bunker is a great place to meet with your colleagues, talk about homework, and get to know some of the people who have gone through Math 10850/10860 before you. Have a look at the article at

`https://www3.nd.edu/~dgalvin1/10850/10850_F19/MathBunker.pdf`

to get a better sense of what the math bunker is about. The times are

- Sunday through Thursday, 7pm-9pm.

Feel free at any time to email me or ask me before or after class for help interpreting unclear language in any question.

A final general note: you have a week to do each homework assignment. Please make use of all that time! Don't put off starting the assignment until Thursday next — read through the problems now, and let your mind mull over them for a while. You might find it helpful to get easier problems out of the way early so you have time to focus on the harder ones.

Writing homework solutions

There are some basic ground rules for homework: points will be deducted from the first homework for not following these rules; subsequent homeworks may be not accepted.

- On the top of the first page, write your name clearly, and write which assignment it is ("Homework 1", "Homework 2", etc.).
- Leave a margin on the page, at least 1/2 inch all around (left, right, top, bottom).
- Present the solutions in order (question 1 first, them question 2 etc.)[1]
- Leave space between solutions to different problems, ideally at least 1 inch. This makes it much easier for the graders to follow, and gives them room to leave comments.
- **Staple all pages together!**

Here are some more general comments:

Please write neatly and clearly. Don't attempt to cram everything you want to write into a little corner of the page; be conscious of the fact that a grader will be reading your assignments, and will not have you nearby for clarification, if something is written too small or too indistinctly. Be aware too that the grader has 40+ scripts to grade, and will deeply appreciate not having to take out a magnifying glass to read your solutions!

If you mess up a problem, please don't cross out large chunks! Just re-start on a fresh page. The same goes for late insertions of new material to a solution. Ideally you should solve each problem on rough notes first, just for yourself, and then when you are sure that you have a good solution, you should present that neatly on a fresh page to turn in.

Part of what we will be learning as the year progresses is the art (art, not science) of good mathematical writing. I won't say much about it here, as it is something we will mostly be learning by experience. What I will say is this: with your solution to each problem, you are aiming to convince a reader that you understand the problem, and that you have a valid solution. The reader is not someone known to you, to whom you can give clarifications if necessary; she is basing her assessment of whether you understand to problem, and have a solution, solely on your written presentation. So you need to explain each step you take,

[1] Note this doesn't mean that you have to *think about* the questions in the order I ask them. Ideally you should be doing rough drafts of all your solutions, in whatever order suits you, and then writing the solutions up more neatly before submitting.

and justify all of your assertions. On the other hand, you need not be overly verbose; you can assume that the reader understands all the mathematics that you are using in your solutions, and so you can focus on clearly presenting your ideas, rather than on re-inventing the "mathematical wheel". There is a delicate balance here between being complete and being concise, and in truth this is an aspect of mathematical writing that takes a lifetime to master.

It is the custom to use complete sentences when writing mathematics, and you should aim to do this as much as possible. Follow the lead of the text book and my class notes to see how formulas and equations can be incorporated into mathematical writing. For example, notice that in the textbook you will see things like "Using Penrose's rule, we known that $A = B$. But we also know that $B = C$ (it was a hypothesis of the problem). So we can conclude that $A = C$", rather than impenetrable chains of unconnected assertions such as "$A = B, B = C, C = A$". Don't worry if this seems vague at first; this is another aspect of mathematical writing that takes a long time to fully master.[2]

Feel free to talk to me during office hours about mathematical writing (I will happily critique a proposed presentation).

A final note on writing: while I encourage you to talk to each other about the homework problems, **the final write-up of each problem should be entirely your own**. If you use any external resources (a solution manual, an internet discussion, a fine-detail discussion with a friend), you should indicate the source clearly in your write-up. Appropriately citing sources is a fundamental principle of scientific ethics!

Reading for this homework

Sections 1 and 2, and Appendix A, of the class notes.

Assignment

1. Carefully and completely read the sections of this document headed "General notes on the homework" and "Writing homework solutions", and ask me if you need any clarification! [This question will not be graded, but you will lose points for failing to follow the basic guidleines presented here.]

2. Do the assigned reading for this weeks' homework [This question also will not be graded, but neglecting to do this question may indirectly lead to a lower homework score.]

3. (a) Verify, using a truth table, that the proposition "not (p implies q)" is equivalent to the proposition "p and (not q)".

   (b) Show that the negation of the proposition $p \Rightarrow q$ (so, the proposition "It is not the case that p implies q") is *not* equivalent to any implication that involves

   - one of p, $\neg p$, q or $\neg q$ on one side
   - one of p, $\neg p$, q or $\neg q$ on the other side

[2] See Section 1.4 of the class notes for more on this, and see Section 2.4 for examples of "proofs presented in prose".

- exactly one occurrence of one of p, $\neg p$ in total, and exactly one occurrence of one of q, $\neg q$ in total.

(There is a slow way to do part (b), by checking all possible cases. That seems like a pain! Can you find a fast way?)

4. (This question provides another justification for the definition/truth table of $p \Rightarrow q$.)

   Suppose we hadn't defined implication, and are trying to "reason out" a truth table for $p \Rightarrow q$. We should all agree that if p is true and q is true then $p \Rightarrow q$ is true (is a valid contract/promise), and that if p is true but q is false then $p \Rightarrow q$ is false (is an invalid contract/promise). The rest of the truth table for $p \Rightarrow q$ is more problematic :(.

   There are four possible ways of completing the truth table, leading to four "candidate" truth tables for $p \Rightarrow q$. Here are the four, which I have chosen to label $p \Rightarrow_1 q$, $p \Rightarrow_2 q$, $p \Rightarrow_3 q$ and $p \Rightarrow_4 q$:

p	q	$p \Rightarrow_1 q$	$p \Rightarrow_2 q$	$p \Rightarrow_3 q$	$p \Rightarrow_4 q$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	T	F	F
F	F	T	F	T	F

   Notice that $p \Rightarrow_1 q$ is the one that agrees with $p \Rightarrow q$, as we have defined it.

   Another aspect of implication that we should all agree on is the fundamental rule of inference, formally referred to as *modus ponens*, which says that if we known that p is true and also that p implies q, then we can infer that q is true. In other words, it should be the case that

$$(p \wedge (p \Rightarrow q)) \Rightarrow q$$

   is a tautology (an absolutely true statement — T's in every row of the final column of the truth table).

   (a) Which of the following are tautologies? (Construct truth tables for each.)

      i. $(p \wedge (p \Rightarrow_1 q)) \Rightarrow_1 q$
      ii. $(p \wedge (p \Rightarrow_2 q)) \Rightarrow_2 q$
      iii. $(p \wedge (p \Rightarrow_3 q)) \Rightarrow_3 q$
      iv. $(p \wedge (p \Rightarrow_4 q)) \Rightarrow_4 q$

   (b) Interpret your answer to part (a).

5. Parentheses matter in general — "$\neg(p \wedge q)$" isn't the same as "$(\neg p) \wedge q$" — but presumably parentheses *don't* matter when a statement is just the "and" of a bunch of simpler statements. Indeed, it's easy to verify, via a truth table, the associative law that

$$p \wedge (q \wedge r) \quad \text{is the same as} \quad (p \wedge q) \wedge r.$$

   What about the "and" of four statement, though? There are *five* different ways that parentheses can be put around $p \wedge q \wedge r \wedge s$, to make different-looking expressions:

A: $((p \wedge q) \wedge r) \wedge s$

B: $(p \wedge (q \wedge r)) \wedge s$

C: $p \wedge ((q \wedge r) \wedge s)$

D: $p \wedge (q \wedge (r \wedge s))$

E: $(p \wedge q) \wedge (r \wedge s)$.

These are all the same, right? Well, it should be pretty clear that A and B are the same: since $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ are the same (by associativity), it must be that $((p \wedge q) \wedge r) \wedge s$ and $(p \wedge (q \wedge r)) \wedge s$ are the same. Similarly, C and D are the same. But it's not immediately obvious that A/B is the same as C/D, or that E is the same as these two.

(a) Check that indeed B and C are the same. You could do this by building a truth table, but that is a drudge (16 rows!). There's a clever way to do it, that involves applying the associative rule in a slightly non-obvious way. Try to find the clever way!

(b) Check that B and E are the same. (Again, there's a slow way, and a clever way. Seek cleverness.)

We will eventually be able to prove that for *any* n, no matter how the "and" of n things is parenthesized, the result is logically the same. For the moment we will just take this for granted.

6. Later in the semester we will discuss the *Archimedean principle* of positive real numbers:

   "If N and s are positive numbers, there's a positive number t with $ts > N$."

   (This is true no matter how big N is or how small s is.)

   (a) If the universe of discourse for all the variables involved (t, s and N) is the set of positive real numbers, then we can encode this statement as

   $$(\forall N)(\forall s)(\exists t)(ts > N).$$

   Write down the *negation* of the Archimedean principle symbolically, pulling the negation through all the quantifiers (your final answer should not involve the symbol "$\neg$"). Interpret the answer in ordinary English, by writing a sentence that captures what precisely it means for the Archimedean principle to *not* be true.

   (b) If the universe of discourse for all the variables involved is the set of *all* real numbers (positive or otherwise), then the encoding of the principle is the slightly more complicated

   $$(\forall N)(\forall s)\left[((N > 0) \wedge (s > 0)) \Rightarrow (\exists t)((t > 0) \wedge (ts > N))\right].$$

   Negate this statement, pulling the negation all the way through all the quantifiers, and the implication. Your final answer should be a symbolic statement that has no "$\neg$" symbol in it, nor any "$\Rightarrow$".

7. There is an equivalence underlying "proof by cases" (see Section 2.4 of the class notes):

$$(p_1 \vee p_2 \vee \cdots \vee p_k) \Rightarrow q \text{ is the same as } (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \cdots \wedge (p_n \Rightarrow q).$$

(Saying: if you want to prove that p implies q, and p breaks into cases, $p_1, p_2, \ldots, p_k$, then what you have to do is show that *each* of $p_1, p_2, \ldots, p_k$ on their own imply q.)

Without using truth tables show that this is a correct equivalence when $k = 2$, that is, show that

$$(p_1 \vee p_2) \Rightarrow q \text{ is the same as } (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q).$$

(What you need to do is use the various pairs of equivalent statements listed on page 18 of the class notes, to create a chain of statements, all of which are equivalent, with $(p_1 \vee p_2) \Rightarrow q$ at the start of the chain and $(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q)$ at the end.[3])

8. I have in mind a certain statement $P = P(p, q, r, s)$ that is made up of four simpler

[3]Here's an example of what I'm thinking of. Suppose you want to show that $(p \wedge (p \Rightarrow q)) \Rightarrow q$ is equivalent simply to T (so, is a tautology), without using a truth table. In other words, you want to "reason out" that modus ponens is a valid logical inference, rather than "brute force" it. Here would be one possible approach:

$(p \wedge (p \Rightarrow q)) \Rightarrow q$	is equivalent to	$\neg(p \wedge (p \Rightarrow q)) \vee q$ (Implication law, or definition of implication)
	which is equivalent to	$((\neg p) \vee \neg(p \Rightarrow q)) \vee q$ (De Morgan's law)
	which is equivalent to	$((\neg p) \vee (p \wedge (\neg q))) \vee q$ (Negation of implication)
	which is equivalent to	$(((\neg p) \vee p) \wedge ((\neg p) \vee (\neg q))) \vee q$ (Distributive law)
	which is equivalent to	$((p \vee (\neg p)) \wedge ((\neg p) \vee (\neg q))) \vee q$ (Commutative law)
	which is equivalent to	$(T \wedge ((\neg p) \vee (\neg q))) \vee q$ (Tautology law)
	which is equivalent to	$(((\neg p) \vee (\neg q)) \wedge T) \vee q$ (Commutative law)
	which is equivalent to	$((\neg p) \vee (\neg q)) \vee q$ (Identity law)
	which is equivalent to	$(\neg p) \vee ((\neg q) \vee q)$ (Associative law)
	which is equivalent to	$(\neg p) \vee (q \vee (\neg q))$ (Commutative law)
	which is equivalent to	$(\neg p) \vee T$ (Tautology law)
	which is equivalent to	T (Domination law).

This seems like overkill: it would have been much faster, *in this particular case*, to use a truth table. But as the number of simpler propositions involved grows, the truth table approach becomes less and less desirable. For example, with two propositions (as we have here) the truth table has only $2^2 = 4$ rows; but with 20 propositions, the truth table has $2^20 \approx 1,000,000$ rows! At that point, it is completely impractical to use a truth table to verify an equivalence, and it is absolutely necessary to the pure reasoning illustrated in the example above, and asked for in this homework problem.

statements, p, q, r and s. Here's the truth table of P:

p	q	r	s	P
T	T	T	T	F
T	T	T	F	F
T	T	F	T	T
T	T	F	F	T
T	F	T	T	F
T	F	T	F	T
T	F	F	T	T
T	F	F	F	F
F	T	T	T	T
F	T	T	F	T
F	T	F	T	F
F	T	F	F	F
F	F	T	T	T
F	F	T	F	F
F	F	F	T	F
F	F	F	F	F

Express P in terms of p, q, r and s, using the symbols $\neg$, $\vee$ and $\wedge$. (This is closely related to Exercise number 5 on the first tutorial handout.)

9. This question concerns the statement $(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s)$. This may seem somewhat random, but it is in fact (one special case of) and incredibly important logical operation in theoretical computer science. I won't say the name commonly given to the operation, as it gives the following question away.

   (a) Write down the truth table of the statement

   $$(p \Leftrightarrow q) \Leftrightarrow (r \Leftrightarrow s).$$

   (b) There is a *very* simple description of what P is doing. Can you find it? Your answer should take the form:

   "P is true exactly when ..."

   and you should not use more than eleven words to replace "..." (with p, q, r and s each counting as one word).

Extra credit problems

A note on extra credit problems: I will not add the scores for these questions to your score for the homework. Instead I will grade them, and make a note on my spreadsheet of how many you answered correctly. I will indicate this with an appropriate number of "+" signs on the front page of your script. Each extra credit problem answered correctly will be worth 1/10 of a homework.

Please give me your extra credit solutions on a separate sheet to the rest of the homework (so I can grade it; the rest of the homework I pass to the graders).

1. Show that whenever 501 different numbers are chosen from among the numbers 1 through 1000, there must be some two of them that have no factors in common (e.g., among the numbers $2, 4, 8, 12, 15$, you could choose 2 and 15, or 4 and 15, or 8 and 15, but not 8 and 12 since they have the factor 4 in common).

2. Show that whenever 501 different numbers are chosen from among the numbers 1 through 1000, there must be some two of them, one of which divides the other.

3. Show that both statements above are false if "501" is replaced by "500".