# Solutions to Exam 3, Math 10560

1. *Compute the following limit:* $\lim_{n\to\infty} \dfrac{\sin n}{n^2}$.

   **Solution:** Note $\dfrac{-1}{n^2} \le \dfrac{\sin n}{n^2} \le \dfrac{1}{n^2}$. Both $\dfrac{-1}{n^2}$ and $\dfrac{1}{n^2}$ tend to zero as $n$ tends to infinity. So by taking the limits of the bounding functions and using the Squeeze Theorem, we get $\lim_{n\to\infty} \dfrac{\sin n}{n^2} = 0$.

2. *Compute the following limit* $\lim_{n\to\infty} \dfrac{3n^2(n-2)!}{n!}$.

   **Solution:** Note

$$\frac{3n^2(n-2)!}{n!} = \frac{3n^2(n-2)!}{n(n-1)(n-2)!} = \frac{3n^2}{n(n-1)} = \frac{3n}{n-1}$$

   so

$$\lim_{n\to\infty} \frac{3n^2(n-2)!}{n!} = \lim_{n\to\infty} \frac{3n}{n-1} = \lim_{n\to\infty} \frac{3}{1-\frac{1}{n}} = 3.$$

3. *Does the series* $\sum_{n=0}^{\infty} \dfrac{3+2^n}{\pi^{n+1}}$ *converge or diverge? If it converges, compute its value.*

   **Solution:** Note

$$\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^{n+1}} = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^n} = \frac{1}{\pi}\sum_{n=0}^{\infty} \left(\frac{3}{\pi^n} + \frac{2^n}{\pi^n}\right) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{3}{\pi^n} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{2^n}{\pi^n}.$$

   The two sums are both geometric series, the first with $a = 3$ and $|r| = |1/\pi| < 1$ and the second with $a = 1$ and $|r| = |2/\pi| < 1$. Hence each series converges and our splitting the series into two was valid. Moreover

$$\frac{1}{\pi}\sum_{n=0}^{\infty} \frac{3}{\pi^n} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{2^n}{\pi^n} = \frac{1}{\pi}\left(\frac{3}{1-\frac{1}{\pi}}\right) + \frac{1}{\pi}\left(\frac{1}{1-\frac{2}{\pi}}\right) = \frac{3}{\pi-1} + \frac{1}{\pi-2}.$$

4. *Which of the following statements are true about the series* $\sum_{n=1}^{\infty} \frac{n^2+1}{n^5-n^2\sqrt{3}}$*?*

   *I. This series converges because* $\lim_{n\to\infty} \dfrac{n^2+1}{n^5-n^2\sqrt{3}} = 0$.

   *II. This series converges by Ratio Test.*

   *III. This series converges by Limit Comparison Test against the p-series* $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$.

**Solution:** Look at each part.

I. Although $\lim\limits_{n\to\infty} \dfrac{n^2+1}{n^5 - n^2\sqrt{3}} = 0$, we cannot conclude anything from this. (This is using the Test for Divergence which is inconclusive here.)

II. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{\frac{(n+1)^2+1}{(n+1)^5-(n+1)^2\sqrt{3}}}{\frac{n^2+1}{n^5-n^2\sqrt{3}}}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^2+2n+2}{n^5+\cdots+1-\sqrt{3}} \cdot \dfrac{n^5-n^2\sqrt{3}}{n^2+1}\right|$ $= \lim\limits_{n\to\infty} \left|\frac{n^7+\cdots-2n^2\sqrt{3}}{n^7+\cdots+1-\sqrt{3}}\right| = 1$; this is the one situation in which the Ratio Test is inconclusive.

III. $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n^3}}{\frac{n^2+1}{n^5-n^2\sqrt{3}}} = \lim\limits_{n\to\infty} \dfrac{n^5-n^2\sqrt{3}}{n^5+n^3} = 1$, so by (limit) comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, the series converges.

Therefore only III is true.

5. *One of the statements below holds for the series* $\sum\limits_{n=1}^{\infty} \dfrac{\cos(2n)}{n^2+1}$. *Which one?*

(a) This series is absolutely convergent by Comparison Test.
(b) This series is conditionally convergent.
(c) This series converges by Alternating Series Test.
(d) This series diverges by Ratio Test.
(e) This series diverges because $\lim\limits_{n\to\infty} \frac{\cos(2n)}{n^2+1}$ is not 0.

**Solution:** Note $|\cos(2n)| < 1$ and $\left|\dfrac{1}{n^2+1}\right| \leq \dfrac{1}{n^2}$ for all $n$; thus $\left|\dfrac{\cos(2n)}{n^2+1}\right| \leq \dfrac{1}{n^2}$ for all $n$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges as a $p$-series with $p = 2 > 1$, so does $\sum\limits_{n=1}^{\infty} \left|\dfrac{\cos(2n)}{n^2+1}\right|$ by the Comparison Test. So the original series converges absolutely, and (a) is true. The remaining statements are false: (b), because if the series were conditionally convergent it would not be absolutely convergent; (c), because although the series has positive and negative terms, it is not alternating, so the alternating series test does not apply; (d), because the ratio test leads to a limit of 1, which is inclusive, and (e), because the series converges, so the limit as $n$ tends to infinity of the $n$th term is 0.

6. *Which of the following statements are true about the series* $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^2}$*?*

*I. This series converges by the Alternating Series Test.*

*II. This series converges by the Ratio Test.*

*III. This series converges absolutely.*

**Solution:**

I. $\frac{1}{n^2}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^2} = 0$ so the Alternating Series Test says that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges.

II. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = 1$, so we get no conclusion from the Ratio Test.

III. Taking the absolute value of $\frac{(-1)^{n-1}}{n^2}$ results in a $p$-series with $p = 2 > 1$, so we conclude that the series converges absolutely.

So I and III are true, II is false.

7. *Compute the radius of convergence of the power series* $\sum_{n=1}^{\infty} 2^n (x-1)^{2n}$.

**Solution:** Using the Root Test $\lim_{n\to\infty} \sqrt[n]{|2^n(x-1)^{2n}|} = \lim_{n\to\infty} |2(x-1)^2| = |2(x-1)^2|$. Now $|2(x-1)^2| < 1 \Rightarrow (|x-1|)^2 < \frac{1}{2} \Rightarrow |x-1| < \frac{\sqrt{2}}{2}$. So $R = \frac{\sqrt{2}}{2}$.
The same conclusion could be reached using the ratio test.

8. *Identify the Taylor Series of* $f(x) = \sin(x)$ *centered at* $a = \frac{\pi}{2}$ *and its interval of convergence.*

**Solution:** Note

$$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f''''(x) = \sin x, ...$$

so

$$f(\frac{\pi}{2}) = 1, f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1, f'''(\frac{\pi}{2}) = 0, ...$$

(with the pattern $1, 0, -1, 0$ repeating).
So the Taylor Series is:

$$1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + ...$$

which in summation notation is $\sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{\pi}{2})^{2n}}{(2n)!}$.

Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n(x-\frac{\pi}{2})^{2n}}{(2n)!}}\right| = \lim_{n\to\infty} \left|\frac{(x-\frac{\pi}{2})^2}{(2n+2)(2n+1)}\right| = 0$$

regardless of the value of $x$, from the Ratio Test we conclude that the series converges for all $x$ (interval of convergence is $(-\infty, \infty)$).

9. *The following is the fourth order Taylor polynomial of the function $f(x)$ at $a$.*

$$T_4(x) = 10 + 5(x-a) + \sqrt{3}(x-a)^2 + \frac{1}{2\pi}(x-a)^3 + 17e(x-a)^4$$

*What is $f'''(a)$?*

**Solution:** By the Taylor formula, we have $\dfrac{f'''(a)}{3!} = \dfrac{1}{2\pi}$ (which is the coefficient of $(x-a)^3$) and hence $f'''(a) = \dfrac{1 \cdot 2 \cdot 3}{2\pi} = \dfrac{3}{\pi}$.

10. a) (5 pts) *Give a power series representation for $e^{x^2}$.*
**Solution:** Since the $n$-th derivative of $e^x$ is $e^x$ and $e^0 = 1$,

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

Substituting $x^2$ for $x$, we have

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n} = 1 + x^2 + \frac{1}{2}x^4 + \cdots.$$

b) (5 pts) *Find the limit*

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^4}.$$

**Solution:**

$$\frac{e^{x^2} - 1 - x^2}{x^4} = \frac{(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots) - 1 - x^2}{x^4} = \frac{\frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots}{x^4} = \frac{1}{2} + \frac{1}{6}x^2 + \cdots.$$

So

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^4} = \lim_{x \to 0} \frac{1}{2} + \frac{1}{6}x^2 + \cdots = \frac{1}{2}.$$

(This part could also be done using L'Hospital's rule, but it would require multiple iterations: $\lim_{x \to 0} \frac{e^{x^2}-1-x^2}{x^4} = \lim_{x \to 0} \frac{2xe^{x^2}-2x}{4x^3} = \lim_{x \to 0} \frac{e^{x^2}-1}{2x^2} = \lim_{x \to 0} \frac{2xe^{x^2}}{4x} = \lim_{x \to 0} \frac{e^{x^2}}{2} = \frac{1}{2}$. )

11. *Consider the function $f(x) = \frac{1}{2-3x}$.*

a) (4 pts.) *Find the Taylor series of $f(x)$ centered at 0.*
**Solution:** We have

$$\frac{1}{2-3x} = \frac{1}{2(1-\frac{3}{2}x)} = \frac{1}{2}\left(\frac{1}{1-\frac{3}{2}x}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{3}{2}x)^n = \sum_{n=0}^{\infty} \frac{3^n}{2^{n+1}}x^n,$$

as long as $|r| = |\frac{3}{2}x| < 1$.

b) (3 pts.) *Determine the radius of convergence of this power series.*

**Solution:** From our knowledge of geometric series, the series converges if and only if $|r| = |\frac{3}{2}x| < 1$, which implies $|x| < \frac{2}{3}$, and so the radius $R$ is $R = \dfrac{2}{3}$. One can also arrive at this conclusion by using the Ratio Test or the Root Test.

c) (4 pts) *Find a power series representation for* $\frac{1}{(2-3x)^2}$ *and give its radius of convergence.*

**Solution:** Differentiating the function in part a) and differentiating term by term the corresponding power series representation found in part a), we get

$$\frac{3}{(2-3x)^2} = \sum_{n=1}^{\infty} \frac{n3^n}{2^{n+1}} x^{n-1}.$$

Dividing both sides by 3, we get

$$\frac{1}{(2-3x)^2} = \sum_{n=1}^{\infty} \frac{n3^{n-1}}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)3^n}{2^{n+2}} x^n.$$

The radius of convergence of this series is $2/3$, the same as that of the series in part a).

You can also obtain the answer by squaring the power series found in part a): $\frac{1}{4}(1 + \frac{3}{2}x + (\frac{3}{2}x)^2 + (\frac{3}{2}x)^3 + \cdots)(1 + \frac{3}{2}x + (\frac{3}{2}x)^2 + (\frac{3}{2}x)^3 + \cdots) = \frac{1}{4}(1 + 2(\frac{3}{2}x) + 3(\frac{3}{2}x)^2 + 4(\frac{3}{2}x)^3 + \cdots)$.

d) (1 pt) *What is the value of the series you found in part (c) at* $x = 1/2$*?*

**Solution:** The value is given by

$$\left.\frac{1}{(2-3x)^2}\right|_{x=\frac{1}{2}} = \frac{1}{(2-3(\frac{1}{2}))^2} = 4.$$

12. *Find the interval of convergence of the following power series:*

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n}.$$

**Solution:** Using the Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{n+1}}{\frac{(x+1)^n}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |x+1| = |x+1|.$$

We want this to be less than one. So $-1 < x+1 < 1$ which implies $-2 < x < 0$. Next, we need to check the end points.

- $x = -2$ : We have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test.
- $x = 0$ : We have $\displaystyle\sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which diverges because it is a $p$-series with $p = 1$.

Hence, the interval of convergence is $[-2, 0)$.

13. *Use the Integral Test to determine whether the series*

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$$

*is divergent or convergent. You must show that the Integral Test can be used in this situation.*
`Note:` *A correct answer with no work is worth only 3 points.*
`Hint:` *Use Integration By Parts.*

**Solution:** Note $\frac{d}{dx}\left(\frac{\ln(x)}{x^3}\right) = \frac{1-3\ln(x)}{x^4}$. This will be negative when the numerator is negative, that is when

$$0 > 1 - 3\ln(x) \Leftrightarrow 3\ln(x) > 1 \Leftrightarrow \ln(x) > \frac{1}{3} \Leftrightarrow x > \sqrt[3]{e}.$$

This holds for $x \geq 2$, since $8 > e$ or $2 > \sqrt[3]{e}$. Therefore, $\frac{\ln(x)}{x^3}$ is positive, decreasing and continuous (since it is differentiable) for $x \geq 2$. We can use the Integral Test. Next, we evaluate $\int_2^{\infty} \frac{\ln(x)}{x^3} dx$. Let $u = \ln(x)$ and $dv = \frac{1}{x^3} dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{x^{-2}}{-2}$. Using Integration by Parts, we get

$$\int \frac{\ln(x)}{x^3} dx = \frac{-\ln x}{2x^2} + \int \frac{1}{2x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$$

So using the definition of an improper integral and L'Hôpital's Rule

$$\begin{aligned} \int_2^{\infty} \frac{\ln(x)}{x^3} dx &= \lim_{t\to\infty} \left(-\frac{\ln t}{2t^2} - \frac{1}{4t^2} + \frac{\ln 2}{8} + \frac{1}{16}\right) = \lim_{t\to\infty} \left(-\frac{\frac{1}{t}}{4t}\right) - 0 + \frac{\ln 2}{8} + \frac{1}{16} \\ &= \frac{\ln 2}{8} + \frac{1}{16}. \end{aligned}$$

The integral converges, therefore the series converges.