SOLUTIONS TO PRACTICE EXAM 3, MATH 10560

1. *Calculate*

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n}.$$

Solution: Use L'Hospital's rule (twice):

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{((\ln x)^2)'}{x'} = \lim_{n\to\infty} \frac{2\frac{1}{x}\ln x}{1} = \lim_{x\to\infty} \frac{(2\ln x)'}{x'} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0.$$

2. *Find* $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3\cdot 5^{n-1}}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{3\cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3\cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4\cdot 4^{n-1}}{3\cdot 5^{n-1}} = \frac{4}{3}\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^{n-1} = \frac{4}{3}\left(\frac{1}{1-\frac{4}{5}}\right) = \frac{20}{3}.$$

(The series is geometric with $a = \frac{4}{3}$ and $r = \frac{4}{5}$.)

3. *Discuss the convergence of the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

Solution: It converges conditionally. It's an alternating series with $b_n = 1/\sqrt{n}$. We have (i) The sequence $\{b_n\}_{n=2}^{\infty}$ is decreasing since $\sqrt{n+1} > \sqrt{n}$ and thus $b_{n+1} = 1/\sqrt{n+1} < 1/\sqrt{n} = b_n$ for all $n \geq 2$. (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$. Thus the series converges by the Alternating Series Test. But the series $\displaystyle\sum_{n=2}^{\infty} |\frac{(-1)^{n+1}}{\sqrt{n}}| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \frac{1}{2} < 1$.

4. *Use Comparison Tests to determine which* **one** *of the following series is divergent.*

Solution: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}+1}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, a p-series with $p = \dfrac{3}{2} > 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+8}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ diverges by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.

(d) $\displaystyle\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ converges since it is a geometric series with $|r| = \dfrac{5}{6} < 1$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{1}{2}\right)^n$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, a geometric series with $|r| = \dfrac{1}{2} < 1$.

5. *Which series below is the MacLaurin series (Taylor series centered at* 0*) for* $\frac{x^2}{1+x}$ *?*
Solution:

$$\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2},$$

for $|x| < 1$.

6. *Find the degree 3 MacLaurin polynomial (Taylor polynomial centered at* 0*) for the function*

$$\frac{e^x}{1-x^2}$$

Solution: $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, and $\dfrac{1}{1-x^2} = \displaystyle\sum_{n=0}^{\infty} x^{2n}$. Thus writing only terms that will contribute, (that is terms of degree 3 or less)

$$\begin{aligned}\frac{e^x}{1-x^2} &= e^x\cdot\frac{1}{1-x^2} = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)(1+x^2+\cdots)\\ &= 1+x^2+x+x^3+\frac{x^2}{2}+\frac{x^3}{6}+\cdots = 1+x+\frac{3}{2}x^2+\frac{7}{6}x^3+\cdots.\end{aligned}$$

So the degree 3 Taylor polynomial is

$$T_3(x) = 1+x+\frac{3}{2}x^2+\frac{7}{6}x^3.$$

7. *Which series below is a power series for* $\cos(\sqrt{x})$ *?*

Solution: Since $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, we have

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty}(-1)^n\frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!} = 1-\frac{x}{2!}+\frac{x^2}{4!}-\cdots.$$

8. *Calculate*

$$\lim_{x\to 0}\frac{\sin(x^3)-x^3}{x^9}.$$

Solution: Since $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have

$$\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots ,$$

and

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots) - x^3}{x^9} = \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} = -\frac{1}{6}.$$

9. *Does the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Solution: Let $a_n = \frac{(n!)^n}{n^{2n}} = (\frac{n!}{n^2})^n$. Since

$$\lim_{n\to\infty} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{n-1}{n} \cdot (n-2)! = \infty,$$

we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \big(\frac{n!}{n^2}\big)^n = \infty.$$

Hence $\lim_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series is divergent.

Another possibility is to use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty.$$

Since the limit is > 1, the series diverges.

10. *Use the Integral Test to discuss whether the series* $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ *converges.*

Solution: Let $f(x) = \frac{(\ln x)^2}{x}$. It is continuous and positive when $x > 1$. Note

$$f'(x) = \frac{2x\ln x \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{2\ln x - (\ln x)^2}{x^2} = \frac{\ln x}{x^2}(2 - \ln x),$$

and $2 - \ln x < 0$, i.e., $2 < \ln x$, if $x > e^2$. Then $f'(x) < 0$ and hence $f(x)$ is decreasing for $x > e^2$. Therefore, we can use the Integral Test. We compute the indefinite integral $\int \frac{(\ln x)^2}{x} dx$ using the substitution $u = \ln x$, getting

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

So
$$\int_1^\infty \frac{(\ln x)^2}{x}dx = \lim_{t\to\infty}\int_1^t \frac{(\ln x)^2}{x}dx = \lim_{t\to\infty}\frac{(\ln x)^3}{3}\bigg|_1^t = \lim_{t\to\infty}\frac{(\ln t)^3}{3} = \infty.$$
Hence the improper integral $\displaystyle\int_1^\infty \frac{(\ln x)^2}{x}dx$ is divergent. By the Integral Test, the series $\displaystyle\sum_{n=1}^\infty \frac{(\ln n)^2}{n}$ diverges. (Note: It is easier to prove this series diverges using the Comparison Test, comparing to the harmonic series. But we were not at liberty to use this test.)

11. *Find the radius of convergence and interval of convergence of the power series*
$$\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}\,(x-3)^n$$

Solution: Set $a_n = \dfrac{(-1)^n}{\sqrt{n}}(x-3)^n$. Using the Ratio Text,
$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}|x-3| = |x-3|.$$
Hence, the radius of convergence is 1, and the series converges absolutely for $|x-3|<1$, or $2<x<4$. For the end points, when $x=2$, the series is $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ which is divergent since it is a p-series with $p=\frac{1}{2}<1$; when $x=4$, the series is $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n(1)^n}{\sqrt{n}}$ which is convergent since it's an alternating series, and $b_n = \frac{1}{\sqrt{n}}$ is decreasing and $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$. (See the solution to Problem #3 for details.) Hence, the interval of convergence is $2<x\le 4$.

12. *(a) Show that*
$$\sum_{n=0}^\infty (-1)^n x^{2n} = \frac{1}{1+x^2}$$
provided that $|x|<1$.
(b) Find
$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

Solution: (a) Since $|x|<1$, we have $|x^2|<1$. Hence
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^\infty (-x^2)^n = \sum_{n=0}^\infty (-1)^n x^{2n}.$$

(b) Integrate both the left and right hands of (a) to get

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C.$$

Letting $x = 0$, we have $C = 0$. Hence, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x.$$

Let $x = \frac{1}{\sqrt{3}}$. We get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$