Mixed Strategy

Recall the case of General Roadrunner and General Coyote; General Roadrunner sends out a bombing sortie consisting of a heavily armed bomber plane and a lighter support plane every day. He can place the bomb on either plane and General Coyote chooses to attack one of the planes. The payoffs shown in the matrix below give the percentage of time the bomb hits the target in each of the four possible situations.

		C. attacks	
		B	S
R.	B	80%	100%
places bomb	S	90%	50%

Mixed Strategy

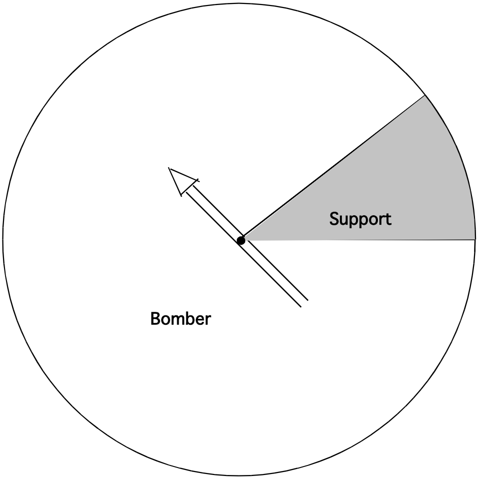
Last time, we saw that this payoff matrix **does not** have a saddle point. If General Roadrunner chooses a fixed strategy of placing the bomb on the bomber each day, General Coyote can reduce the percentage of time the bomb hits the target to 80% by attacking the bomber each day. Therefore General Roadrunner might want to bluff sometimes and put the bomb on the support plane. This would confuse General Coyote who would now have to decide whether to attack the support plane or the bomber plane.

Mixed Strategy

The mathematical model for this type of situation assumes that General Roadrunner does something like putting the bomb on the bomber 90% of the time and on the support plane 10% of the time. He makes his decision of where to put the bomb on each day by using a device such as a spinner shown on the next slide, so that his decisions are not predictable. (More realistically, he uses a random number generator on a calculator, computer programming language, or spreadsheet)

Coyote is then faced with the task of coming up with the best counterstrategy, having deduced Roadrunner's strategy by, for example, observations of repeated plays of the game. We will discuss this in more detail, after mastering calculating with mixed strategies.

A randomness generator



Mathematics of mixed strategies

Example: Consider a zero-sum game with pay-off matrix for R given by

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

A possible *mixed* strategy for R is to play $\begin{bmatrix} .8 & .2 \end{bmatrix}$, which means that R plays $R1$ with probability .8 and $R2$ with probability .2.

We're assume that R and C play the game repeatedly, and that on each play each can choose their play with the aid of some random device. So "R plays $R1$ with probability .8" means that R plays $R1$ 80% of the time, unpredictably.

For C, playing $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ means that C plays $C1$ with probability .3 and $C2$ with probability .7.

Mathematics of mixed strategies

If we let X denote R's payoff for this game, we see that on each play, the random variable X has 4 possible outcomes. We can calculate the expected value of X using the probability distribution of X. For this game, if R plays $\begin{bmatrix} .8 & .2 \end{bmatrix}$ and C plays $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ the probability distribution for X is given by the following table.

Choice	X = Pay-off for R	Probability	$XP(X)$
$R1C1$	-1	$(.8)(.3) = .24$	$-1 \times .24$
$R1C2$	3	$(.8)(.7) = .56$	$3 \times .56$
$R2C1$	2	$(.2)(.3) = .06$	$2 \times .06$
$R2C2$	-2	$(.2)(.7) = .14$	$-2 \times .14$
			$\mu = 1.28$

Mathematics of mixed strategies

Note that in calculating the probabilities here we are assuming that the **players' choices are independent of each other**. So neither player has knowledge of what the other player is choosing. This is the reason that we multiply the probabilities. For example in calculating the probability that R will play $R1$ and C will play $C1$ (and R will lose \$1, a pay-off of -1), we multiply .8, the probability that R will play $R1$, by .2, the probability that C will play $C1$.

The expected pay-off for R is 1.28. This gives the average pay-off for R, if R and C play this game many times with the given strategies. Because it is a zero-sum game the expected pay-off for C and the expected pay-off for R must add to zero, since R's gain is always C's loss. So **C's expected pay-off here is -1.28.**

Mathematics of mixed strategies

If

- R has two options, R1 and R2,
- C has two options, C1 and C2,
- the payoff matrix is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

- R plays the mixed strategy $\begin{bmatrix} r_1 & r_2 \end{bmatrix}$ and
- C plays $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$,

then the expected pay-off for R is

$$[r_1 p_{11} c_1 + r_1 p_{12} c_2 + r_2 p_{21} c_1 + r_2 p_{22} c_2].$$

We write this in the following way:

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Example

Suppose General Roadrunner puts the bomb on the bomber 90% of the time and on the support plane 10% of the time. Suppose also that General Coyote attacks the bomber 50% of the time and attacks the support plane 50% of the time. What is the expected percentage of successful bombing missions? (The payoff matrix is below)

$$\begin{bmatrix} 80 & 100 \\ 90 & 50 \end{bmatrix}$$

$$\begin{bmatrix} .9 & .1 \end{bmatrix} \begin{bmatrix} 80 & 100 \\ 90 & 50 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix} =$$

$[(.9)80(.5) + (.9)100(.5) + (.1)90(.5) + (.1)50(.5)] = [88]$, so 88 is R's payoff; the missions are on average successful 88% of the time. Because this is a constant sum game, with sum 100, the expected payoff for C is $100 - 88 = 12$

Example

R and C play a zero–sum game with Pay-off matrix for R given by

$$\begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}.$$

If R plays a strategy $\begin{bmatrix} .4 & .6 \end{bmatrix}$ and C plays $\begin{bmatrix} .2 \\ .8 \end{bmatrix}$,

(a) What is the expected payoff for R?

$$\begin{bmatrix} .4 & .6 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} .2 \\ .8 \end{bmatrix}$$

$$[(.4) - 2(.2) + (.4)1(.8) + (.6)2(.2) + (.6) - 2(.8)] = [-.56]$$

So R expects to lose .56 on average per game with these mixed strategies.

Example

(b) What is the expected payoff for C?

$0 - (-.56) = .56$ (This is a zero-sum game; C's payoff is the negative of R's payoff. In a constant-sum game with constant K, C's payoff would be $K-$ R's payoff. In the Roadrunner example, $K = 100$)

(c) (Different counterstrategy) If C changes strategy to $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$, what happens to C's expected pay-off?

$\begin{bmatrix} .4 & .6 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = [-.44]$ so $-.44$ is R's payoff and $.44$ is C's payoff.

So if C changes strategy C's payoff is less (Different mixed strategies potentially lead to different payoffs).

Expected payoffs when there are more options

Suppose that in a zero-sum game, R has m options and C has n options.

We can represent a mixed strategy for R by a row of numbers with m entries

$$[r_1, r_2, r_3, \ldots, r_m],$$

where r_i gives the probability that R will play option Ri (Row i); so $0 \le r_i \le 1$, $r_1 + r_2 + r_3 + \cdots + r_m = 1$.

We can represent C's strategy by a column matrix:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where c_i gives the probability that C will play option Ci (Col i), so $0 \le c_i \le 1$, $c_1 + c_2 + c_3 + \cdots + c_n = 1$.

Expected payoffs when there are more options

Suppose the payoff matrix is an array P with m rows (one for each of R's options) and n columns, with the entry in the ith row and jth column being p_{ij} (this is the payoff if R plays Ri and C plays Cj).

Then the **expected pay-off for** R is given by adding up, over all possible choices for i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$, (the payoff when i plays Ri and C plays Cj) times (the probability that those choices are made); that is, we add up the mn expressions of the form $r_i p_{ij} c_j$.

We write this as

$$\left[\ r_1, r_2, \ldots, r_m\ \right] P_{m \times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mu]$$

Expected payoffs when there are more options

Suppose the payoff matrix is an array P with m rows (one for each of R's options) and n columns, with the entry in the ith row and jth column being p_{ij} (this is the payoff if R plays Ri and C plays Cj).

Then the **expected pay-off for** R is given by

$$\begin{array}{ccccccccccc}
 & r_1p_{11}c_1 & + & r_1p_{12}c_2 & + & r_1p_{13}c_3 & + & \cdots & + & r_1p_{1n}c_n \\
+ & r_2p_{21}c_1 & + & r_2p_{22}c_2 & + & r_2p_{23}c_3 & + & \cdots & + & r_2p_{2n}c_n \\
+ & r_3p_{31}c_1 & + & r_3p_{32}c_2 & + & r_3p_{33}c_3 & + & \cdots & + & r_3p_{3n}c_n \\
+ & \cdots & & & & & & & & \\
+ & r_mp_{m1}c_1 & + & r_mp_{m2}c_2 & + & r_mp_{m3}c_3 & + & \cdots & + & r_mp_{mn}c_n.
\end{array}$$

Example

If the payoff matrix for R for a zero-sum game is given by

$$\begin{bmatrix} 2 & -3 \\ 0 & 2 \\ -5 & 10 \end{bmatrix},$$

and R's strategy is given by $[.2\ .5\ .3]$, and C's strategy is given by $\begin{bmatrix} .7 \\ .3 \end{bmatrix}$,

(a) what is the expected pay-off for R?

$\begin{bmatrix} 0.2 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = [0.25]$ so R's payoff is 0.25.

(b) What is the expected pay-off for C for this game?

-0.25

Example

Example (Two Finger Morra) Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player's call matches the total, no money changes hands.

The pay-off matrix for this game is given by:

$$\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$$

Example

(a) If Ruth chooses all of her options with equal probability and Charlie chooses each of his four options with equal probability, what are the expected winnings for Ruth?

$$\begin{bmatrix} .25 & .25 & .25 & .25 \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix} \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} = [0] \text{ so}$$

Ruth's expected winnings are 0.

(b) If Ruth chooses all of her options with equal probability and Charlie sticks to a pure strategy of playing Col. 2 on every play, what are the expected winnings for Ruth?

$$\begin{bmatrix} .25 & .25 & .25 & .25 \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [-0.25].$$

Example

(c) If Ruth always chooses all of her options with equal probability, which of the above two strategies is the better one for Charlie?

With his strategy from part (a), Charlie on average breaks even; with his strategy from part (b), he on average gains .25. So (b) is the better option.

Old Exam Questions

1 Rasputin (R) and Catherine (C) play a zero-sum game with payoff matrix for Rasputin given below. If Rasputin's strategy is given by $[.3, .2, .5]$ and Catherine's strategy is given by $\begin{bmatrix} .2 \\ .1 \\ .7 \end{bmatrix}$, what is the expected pay-off for Rasputin?

	C_1	C_2	C_3
R_1	1	0	0
R_2	1	5	0
R_3	3	2	-2

(a) -0.1 (b) 1 (c) 1.54 (d) -0.3 (e) 0.21

$$\begin{bmatrix} .3 & .2 & .5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} .2 \\ .1 \\ .7 \end{bmatrix} = [-0.1]$$

Old Exam Questions

2 Rusty (R) and Crusty (C) play a zero-sum game with pay-off matrix for Rusty given by:

	$C1$	$C2$
$R1$	2	1
$R2$	5	-1

If Crusty always plays C2, which of the following mixed strategies gives the highest expected pay-off for Rusty?

(a) (.2 .8) (b) (1 0) (c) (0 1) (d) (.5 .5) (e) (.7 .3)

Suppose Rusty choose to play R1 with probability p. Then he plays R2 with probability $1-p$ so Rusty's payoff is

$$\begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1\ 1-p] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1+(-1)(1-p)] = [p]$$

Looking at the first coordinates of the possible answers, we see (b) has the largest value.