Combinations

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Here (from last time) is the list of all the ways of choosing three people, when the order of selection matters:

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

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Now order of selection *doesn't* matter — AMC is the same as ACM — so these sixty possibilities bunch up in groups of 6, with everything in the same group being the same. That leaves 10 different possibilities:

AMC	AMS	AMR	ACS	ACR
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With order mattering, there were $\mathbf{P}(5,3)$ possibilities. With order not mattering, we have overcounted by a factor of $6 = 3!$ (one for each of the ways or putting an order on three people), so the right count is

$$\frac{60}{3!} = \frac{\mathbf{P}(5,3)}{3!} = \frac{5!}{2!3!}.$$

Combinations

We have listed all **Combinations** of the five friends taken 3 at a time. The number of such combinations is denoted by $\mathbf{C}(5,3)$.

This is the same as listing all the subsets of size 3 of the set $\{A, C, M, R, S\}$

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Definition A **Combination** of **n** objects taken **r** at a time is a selection of r objects taken from among the n. The order in which the objects are chosen does not matter.

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Definition A **Combination** of **n** objects taken **r** at a time is a selection of r objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n, r)$ or $\binom{n}{r}$. We have

$$\mathbf{C}(n, r) = \binom{n}{r} = \frac{\mathbf{P}(n, r)}{r!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-r+1)}{r \cdot (r-1) \cdot (r-2) \cdots 1} = \frac{n!}{r! \cdot (n-r)!}$$

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Example Evaluate $\mathbf{C}(10, 3)$.

$$\mathbf{C}(10, 3) = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

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$$\mathbf{C}(40, 7) = \frac{40!}{7! \cdot 33!} = 18,643,560$$

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Example How many ways are there to choose 7 people from a class of 40 students in order to make a team for Bookstore Basketball?

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Example In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

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Example In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

$$\mathbf{C}(15,2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

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$$\mathbf{C}(13, 5) = \frac{13!}{5! \cdot 8!} = 1,287$$

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There are 4 kings and 4 queens. We can select 2 kings in $\mathbf{C}(4, 2)$ ways and we can select 3 queens in $\mathbf{C}(4, 3)$ ways. We can distinguish kings from queens so the answer is $\mathbf{C}(4, 2) \cdot \mathbf{C}(4, 3) = 6 \cdot 4 = 24$.

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

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$\mathbf{C}(30, 5) = 142,506.$

Problems using a mixture of counting principles

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There are $\mathbf{C}(4,2)$ ways to get 2 kings and $\mathbf{C}(48,3)$ ways to fill out the hand. Hence there are $\mathbf{C}(4,2) \cdot \mathbf{C}(48,3)$ hands with exactly 2 kings. There are $\mathbf{C}(4,3) \cdot \mathbf{C}(48,2)$ hands with exactly 3 kings and there are $\mathbf{C}(4,4) \cdot \mathbf{C}(48,1)$ hands with exactly 4 kings. Hence there are
$\mathbf{C}(4,2) \cdot \mathbf{C}(48,3) + \mathbf{C}(4,3) \cdot \mathbf{C}(48,2) + \mathbf{C}(4,4) \cdot \mathbf{C}(48,1)$
hands with at least two kings. The number is
$6 \cdot 17,296 + 4 \cdot 1,128 + 1 \cdot 48 = 108,336.$

Problems using a mixture of counting principles

Example In the Notre Dame Juggling club, there are 5 graduate students and 7 undergraduates. Student Activities will fund 5 people to attend, as long as at least three are undergraduates. In how many ways can 5 people be chosen to go to the performance so that the funding will be granted?

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Break the problem up, by number of undergraduates chosen to attend. Three undergraduates: $\mathbf{C}(7,3) \cdot \mathbf{C}(5,2)$; Four undergraduates: $\mathbf{C}(7,4) \cdot \mathbf{C}(5,1)$; Five undergraduates: $\mathbf{C}(7,5) \cdot \mathbf{C}(5,0)$. The number is $35 \cdot 10 + 35 \cdot 5 + 21 \cdot 1 = 546$.

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Remark: $\mathbf{C}(7,3) \cdot \mathbf{C}(9,2) = 1,260$. Why is this NOT the right answer?

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Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

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Pat's choices are independent so
$\mathbf{C}(3,1) \cdot \mathbf{C}(2,1) \cdot \mathbf{C}(4,2) \cdot \mathbf{C}(3,2) = 3 \cdot 2 \cdot 6 \cdot 3 = 108.$

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$\mathbf{C}(5,4) + \mathbf{C}(5,5).$

Special Cases and Formulas

- It is immediate from the formula $\mathbf{C}(n,k) = \frac{n!}{k!(n-k)!}$ that $\mathbf{C}(n,k) = \mathbf{C}(n,n-k)$ — choosing k things from a set on n to be "in" is the same as choosing $n-k$ things to be "out"

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How many subsets does a set have?

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A set of size n has 2^n subsets — we can choose a subset by going through each element in turn, and deciding whether it is in the subset of not. By the multiplication principle this experiment has $2 \times 2 \times \ldots \times 2 = 2^n$ possible outcomes.

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Also, a set of size n has $\mathbf{C}(n, 0)$ subsets of size 0, $\mathbf{C}(n, 1)$ subsets of size 1, $\mathbf{C}(n, 2)$ subsets of size 2, etc., so $\mathbf{C}(n, 0) + \mathbf{C}(n, 1) + \mathbf{C}(n, 2) + \ldots + \mathbf{C}(n, n-1) + \mathbf{C}(n, n)$ subsets in all

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$C(10,2) + \mathbf{C}(10,3) + \mathbf{C}(10,4) + \mathbf{C}(10,5) + \mathbf{C}(10,6) + \mathbf{C}(10,7) + \mathbf{C}(10,8) + \mathbf{C}(10,9) + \mathbf{C}(10,10).$

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$$2^{10} - \big(\mathbf{C}(10,0) + \mathbf{C}(10,1)\big) = 1024 - (1+10) = 1013$$

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Example How many tips could you leave at a restaurant, if you have a half-dollar, a one dollar coin, a two dollar note and a five dollar note?

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Example How many tips could you leave at a restaurant, if you have a half-dollar, a one dollar coin, a two dollar note and a five dollar note?

You can leave any subset of your money. You have 4 items so there are $2^4 = 16$ possibilities.

The Binomial Theorem

How does this pattern continue?

- $x + y = x + y$
- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

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The Binomial theorem says that for any positive integer n and any two numbers x and y, we have

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

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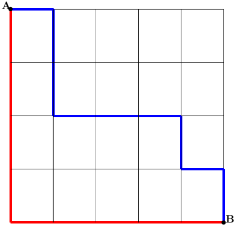
Example If I fully multiply out $(x+y)^{11}$, what's the term involving x^4y^7? From the binomial theorem it is $\binom{11}{4} = 330$.

Taxi Cab Geometry revisited

Recall that the number of taxi cab routes (always traveling south or east) from A to B is the number of different rearrangements of the sequence SSSSEEEEE which is

$$\frac{9!}{4!5!} = \mathbf{C}(9,4) = \mathbf{C}(9,5).$$

Sequences SSSSEEEEE (in red) and ESSEEEESES (in blue) are shown below.



The number of routes equals the number of ways to choose 4 objects from a set of 9 objects, because to determine a route we can start with nine blank slots, and pick 4 of them to be "S"'s (then the remaining 5 slots are forced to be "E"'s)