A puzzle about using information

I have two quarters. One is fair (comes up heads 50% of the time). The other is biased (comes up heads 75% of the time). I choose one of the two coins at random, so it is 50% likely to be the biased coin.

I toss the coin five times, and get the sequence HHHHT. Now how likely is it that I chose the biased coin?

Solution

There are two ways to get HHHHT:

- either I choose the fair coin and toss HHHHT, probability $1/2$ (for choosing the fair coin) times $(1/2)^5$ (for getting HHHHT with fair coin); so overall probability $1/64$
- or I choose the biased coin and toss HHHHT, probability $1/2$ (for choosing the biased coin) times $(3/4)^4(1/4)$ (for getting HHHHT with biased coin); so overall probability $81/2048$.

The conditional probability of choosing biased coin, given that HHHHT is tossed, is the ratio of the probability of choosing biased coin and tossing HHHHT, over the probability of tossing HHHHT, or

$$\frac{81/2048}{81/2048 + 1/64} \approx .717.$$